Day 5: Multiplicity along points of a radicial covering of a regular variety

D. Sulca and O. Villamayor

June 18, 2019





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Fact: The multiplicity along points of X is at most d.

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$$\downarrow \delta \text{ finite} \qquad \cup \qquad \cup$$

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Goal: Study $F_d(X) = \{x \in X : \text{mult}_X(x) = d\}$ (*d*-**fold points**)



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 (d-fold points)

 δ is called transversal if $F_d(X) \neq \emptyset$



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Tranversal morphisms can be blown-up along permissible centers:

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 δ_1 finite of generic rank d.



Summarizing, there is a one to one correspondence between sequences over \boldsymbol{X}

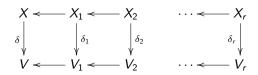
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which in turn can be read off from

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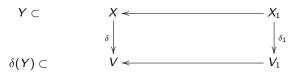
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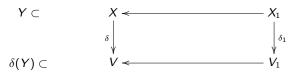
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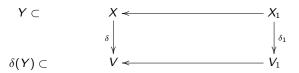


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Preservation under permissible blow-ups:



$$\delta(F_d(X)) = \operatorname{Sing}(J, b)$$
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where $J\mathcal{O}_{V_1}=I(H_1)^bJ_1$, with $H_1\subset V_1$ the exceptional hypersurface.



Roll in resolution of singularities

Goal: Define a sequence of blow-ups along regular centers included in the set of *d*-fold points

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_r$$

with

$$d=\mathsf{max}\,\mathsf{mult}_X=\mathsf{max}\,\mathsf{mult}_{X_1}=\cdots=\mathsf{max}\,\mathsf{mult}_{X_{r-1}}>\mathsf{max}\,\mathsf{mult}_{X_r}$$
 .

This is equivalent to defining a sequence of blow-ups

$$V \longleftarrow V_1 \longleftarrow V_2 \qquad \cdots \longleftarrow V_r$$

where we set $(J_0, b) := (J, b)$ as before, and for i > 0,

- $V_{i-1} \leftarrow V_i \supset H_i$ blow-up along a regular $Z_{i-1} \subset \operatorname{Sing}(J_{i-1}, b)$
- $J_{i-1}\mathcal{O}_{V_i} = I(H_i)^b J_i$
- $Sing(J_r, b) = \emptyset$.



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Summarizing: Radicial morphisms $X \to V$ are viewed from \mathcal{O}_V^q -modules $\mathscr{M} \subset \mathcal{O}_V$.

First description of $\delta(\overline{F_d(X)})$

В

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Corollary

For an \mathcal{O}_V^q -module $\mathscr{M}\subset \mathcal{O}_V$ with associated morphism $\delta:X\to V$ we have

$$\delta(F_d(X)) = \{x \in V : \mathscr{M}_x \subseteq \mathscr{O}_{V,x}^q + m_{V,x}^q\}$$

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$$\mathcal{A}_q := \{(\alpha, \beta) \in \mathbb{N}_0^{r+s} : 0 \le \alpha_i, \beta_i < q\} \supset \mathcal{A}_q^+ := \mathcal{A}_q - \{(0, 0)\}$$



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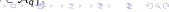
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 \Rightarrow R is free R^q -module with basis $\{\mathbf{x}^{\alpha}\mathbf{y}^{\beta}: (\alpha,\beta) \in \mathcal{A}_q\}$.



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 - $Z \subset \delta(F_d(X))$ (Z is permissible for δ).

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Corollary

- 1) If $\mathfrak{p} \subset R$ is a regular prime, then $R^q + \mathfrak{p}^n = R \cap (R^q_{\mathfrak{p}} + \mathfrak{p}^n R_{\mathfrak{p}})$.
- 2) For an \mathcal{O}_{V}^{q} -module $\mathcal{M} \subset \mathcal{O}_{V}$ and $Z \subset V$ regular, t.f.a.e.:
 - $Z \subset \delta(F_d(X))$ (Z is permissible for δ).
 - $\mathcal{M}_{\xi} \subseteq \mathcal{O}_{V,\xi}^q + m_{V,\xi}^q$, where $\xi \in Z$ is the generic point.
 - $\mathcal{M} \subset \mathcal{O}_V^q + \mathcal{I}(Z)^q$.



 ${\it R}$ an arbitrary ring

 $\mathsf{Hom}_{\mathbb{Z}}(R,R)$

$$\mathsf{Hom}_R(R,R)=R$$

$$\subset \operatorname{\mathsf{Hom}}_{\mathbb{Z}}(R,R)$$

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Differential operators

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There are differential operators $D_{\alpha,\beta}$ for all $(\alpha,\beta) \in \mathbb{N}_0^{r+s}$, s.t.

$$D_{\alpha,\beta}(\mathbf{x}^{\gamma}\mathbf{y}^{\delta}) = {\gamma \choose \alpha} {\delta \choose \beta} \mathbf{x}^{\gamma-\alpha}\mathbf{y}^{\delta-\beta}, \ \forall (\gamma,\delta) \in \mathbb{N}_0^{r+s}$$

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- $\mathrm{Diff}_{R,\mathfrak{m}}^k$ is generated by $\{\mathbf{x}^{\gamma}D_{\alpha,\beta}: |\gamma|=|\alpha|, |\alpha|+|\beta|\leq k\}.$

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- Taylor operators of order q-1 are respectful with q-expansions: For $f=\sum_{\mathcal{A}_q} c_{\alpha,\beta}^q \mathbf{x}^\alpha \mathbf{y}^\beta \in R$, and $D_{\gamma,\delta}$ with $|\gamma|+|\delta|< q$,

$$egin{aligned} D_{\gamma,\delta}(f) &= \sum_{\mathcal{A}_q} c_{lpha,eta}^qinom{lpha}{\gamma}inom{eta}{\delta}\mathbf{x}^{lpha-\gamma}\mathbf{y}^{eta-\delta} \ \mathbf{x}^{\gamma}D_{\gamma,\delta}(f) &= \sum_{\mathcal{A}_a} c_{lpha,eta}^qinom{lpha}{\gamma}inom{eta}{\delta}\mathbf{x}^{lpha}\mathbf{y}^{eta-\delta} \end{aligned}$$

are q-expansions.

Let (R, \mathfrak{m}) be an F-finite regular local ring.

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- ② $f \in R^q$ if and only if $\text{Diff}_{R,+}^{q-1}(f) = 0$.
- $f \in \mathbb{R}^q + \mathfrak{m}^n$ if and only if $\operatorname{Diff}_{R,\mathfrak{m},+}^{q-1}(f) \subseteq \mathfrak{m}^n$.
- If $\nu_{\mathfrak{m}}(f) = qa + b$ with 0 < b < q, then $\nu_{\mathfrak{m}}(D_{\alpha,0}(f)) = qa$ for some α with $|\alpha| = b$.

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- If $\nu_{\mathfrak{m}}(f) = qa + b$ with 0 < b < q, then $\nu_{\mathfrak{m}}(D_{\alpha,0}(f)) = qa$ for some α with $|\alpha| = b$.
- $f \in R^q + \mathfrak{m}^{qa}$ if and only if $\operatorname{Diff}_{R,+}^{q-1}(f) \subseteq \mathfrak{m}^{q(a-1)+1}$. In particular, $f \in R^q + \mathfrak{m}^q$ if and only if $\operatorname{Diff}_{R,+}^{q-1}(f) \subseteq \mathfrak{m}$.

Moving to the global setting

Fix an \mathcal{O}_V^q -module $\mathscr{M}\subset \mathcal{O}_V$ and an ideal $\mathcal{J}\subset \mathcal{O}_V$.

Then there are coherent ideals on V

$$\begin{array}{l} \operatorname{Diff}^0_V(\mathcal{J}) \subset \operatorname{Diff}^1_V(\mathcal{J}) \subset \operatorname{Diff}^2_V(\mathcal{J}) \subset \cdots \\ \operatorname{Diff}^1_{V,+}(\mathcal{M}) \subset \cdots \subset \operatorname{Diff}^{q-1}_{V,+}(\mathcal{M}) \end{array}$$

s. t. for all $x \in V$,

$$(Diff_V^k(\mathcal{J}))_X = Diff_{\mathcal{O}_{V,X}}^k(\mathcal{J}_X)$$

 $(Diff_{V,+}^k(\mathscr{M}))_X = Diff_{\mathcal{O}_{V,X},+}^k(\mathscr{M}_X), \ k = 1, \dots, q-1.$

- ② If $\delta: X \to V$ is the \mathcal{O}_V^q -module associated with \mathcal{M} , then

$$\delta(F_d(X)) = \mathcal{V}(Diff_{V,+}^{q-1}(\mathscr{M})).$$

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$$\begin{split} \nu_x^{(q)}(\mathcal{M}) &:= \max\{n \in \mathbb{N} : \mathcal{M}_x \subset \mathcal{O}_{V,x}^q + m_{V,x}^n\} \text{ (order after cleaning)} \\ \eta_x(\mathcal{M}) &:= \min\{\nu_x(\textit{Diff}_{V,+}^1(\mathcal{M})) + 1, \dots, \nu_x(\textit{Diff}_{V,+}^{q-1}(\mathcal{M})) + q - 1\} \end{split}$$

 $\eta_{\mathsf{x}}(\mathscr{M})$ defines an upper-semicontinuos function $V \to \mathbb{N}$.

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$$\eta_{x}(\mathscr{M}) := \min\{\nu_{x}(\textit{Diff}_{V,+}^{1}(\mathscr{M})) + 1, \dots, \nu_{x}(\textit{Diff}_{V,+}^{q-1}(\mathscr{M})) + q - 1\}$$

 $\eta_{\times}(\mathscr{M})$ defines an upper-semicontinuos function $V \to \mathbb{N}$.

Proposition

 $\nu_x^{(q)}(\mathcal{M}) \leq \eta_x(\mathcal{M})$, and the equality holds if $q \nmid \nu_x^{(q)}(\mathcal{M})$. In addition,

$$\lfloor \frac{\eta_{\mathsf{x}}(\mathscr{M})}{q} \rfloor = \lfloor \frac{\nu_{\mathsf{x}}^{(q)}(\mathscr{M})}{q} \rfloor$$

In particular, $\{x \in V : \mathcal{M}_x \subseteq \mathcal{O}_{V,x}^q + m_{V,x}^{qa}\} = \{x \in V : \eta_x(\mathcal{M}) \ge qa\}.$

Lemma

For a regular hypersurface $H \subset V$,

$$\nu_H(Diff_{V,+}^{q-1}(\mathscr{M})) = qa$$

where
$$a := \max\{a \geq 0 : \mathscr{M} \subseteq \mathscr{O}_V^q + \mathcal{I}(H)^{qa}\} = \lfloor \frac{\eta_H(\mathscr{M})}{q} \rfloor$$
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Assume $a \ge 1$. Then

$$\mathcal{O}_{V}^{q} + \mathscr{M} = \mathcal{O}_{V}^{q} + (\mathcal{O}_{V}^{q} + \mathscr{M}) \cap \mathcal{I}(H)^{qa} = \mathcal{O}_{V}^{q} + (\mathcal{I}(H)^{(q)})^{a} \mathscr{M}_{a}$$

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Lemma

For $\delta:X\to V$ the morphism defined by \mathscr{M} , there are finite morphisms

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \ldots \longleftarrow X_a$$

$$\downarrow \delta \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

that are blow-ups along regular centers $Y_i \subset F_d(X_i)$ whose image in V is H. In addition, the composition $X_a \to V$ corresponds to \mathcal{M}_a .

Let $\mathcal{M} \subset \mathcal{O}_V$ be an \mathcal{O}_V^q -module, and \mathcal{J} an \mathcal{O}_V -ideal.

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• For i = 1, ..., q - 1, there are inclusions

$$\begin{split} I(H_1)^i (\textit{Diff}_{V,+}^i(\mathscr{M}) \mathcal{O}_{V_1}) \subseteq \textit{Diff}_{V_1,H_1,+}^i(\mathscr{M} \mathcal{O}_{V_1}^q) \\ \subseteq \textit{Diff}_{V_1,+}^i(\mathscr{M} \mathcal{O}_{V_1}^q) \subseteq \textit{Diff}_{V,+}^i(\mathscr{M}) \mathcal{O}_{V_1} \end{split}$$

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For any i, there are inclusions

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Lemma (Fundamental point-wise inequality)

If $\nu_x(\mathcal{J})=b$ for all $x\in Z$, so that $\mathcal{JO}_{V_1}=\mathcal{I}(H_1)^b\mathcal{J}_1$ for an ideal \mathcal{J}_1 on V_1 , then

$$u_{\pi(\mathsf{x}_1)}(\mathcal{J}) \ge \nu_{\mathsf{x}_1}(\mathcal{J}_1), \quad \forall \mathsf{x}_1 \in V_1.$$



Let $V \xleftarrow{\pi} V_1 \supset H_1$ be a blow-up along a regular center $Z \subset V$, and set

$$\eta_Z(\mathcal{M}) = qa + b$$
, with $0 \le b < q$, and assume $a \ge 1$.

Then

$$\nu_{H_1}(\operatorname{Diff}_{V_1,+}^{q-1}(\mathscr{MO}_{V_1}^q)) = qa,$$

so that for an $\mathcal{O}_{V_1}^q$ -module \mathcal{M}_a , we have

$$\mathcal{O}_{V_1}^q + \mathscr{M} \mathcal{O}_{V_1}^q = \mathcal{O}_{V_1}^q + (\mathcal{I}(H_1)^{(q)})^{a} \mathscr{M}_{a}$$

In addition, there exists a commutative diagram

$$X \leftarrow \frac{\pi_{\delta}}{\delta} (X \times_{V} V_{1})_{red} \leftarrow \frac{f_{1}}{\delta} \qquad X_{1} \leftarrow \frac{f_{2}}{\delta} \qquad \cdots \leftarrow \frac{f_{a}}{\delta} \qquad X_{a}$$

$$\downarrow^{\delta} \qquad \qquad \downarrow^{\delta_{\pi}} \qquad \qquad \downarrow^{\delta_{1}} \qquad \qquad \downarrow^{\delta_{1}} \qquad \qquad \downarrow^{\delta_{a}}$$

$$V \leftarrow \frac{\pi}{\delta} \qquad V_{1} \leftarrow \frac{1}{\delta} \qquad V_{1} \leftarrow \frac{1}{\delta} \qquad \cdots \leftarrow \frac{1}{\delta} \qquad M.$$

$$M = M \mathcal{O}_{V_{1}}^{q} \qquad M_{1} \qquad M.$$

with $\pi_{\delta} f_1$ the blow-up of X along the center $Y \subset F_d(X)$ s.t. $Z = \delta(Y)$.



q-differential collection

A q-differential collection on V is a sequence of ideals

$$\mathcal{G} = (\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_{q-1})$$

such that $Diff_V^i(\mathcal{J}_j) \subseteq \mathcal{J}_{i+j}$.

Example: for an \mathcal{O}_V^q -module $\mathscr{M}\subset \mathcal{O}_V$,

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We associate with a q-differential collection the function

$$\eta_{\mathsf{x}}(\mathcal{G}) = \min\{\nu_{\mathsf{x}}(\mathcal{J}_1) + 1, \dots, \nu_{\mathsf{x}}(\mathcal{J}_{q-1}) + q - 1\}.$$

This defines an upper-semicontinuous function $V \to \mathbb{N}$.



Transformation of *q*-differential collections

Fix a blow-up $V \xleftarrow{\pi} V_1 \supset H_1$ along a regular center $Z \subset V$.

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Let $\mathcal{G} = (\mathcal{J}_1, \dots, \mathcal{J}_{q-1})$ be a q-differential collection. We denote

$$\mathcal{GO}_{V_1} = (\mathcal{J}_1\mathcal{O}_{V_1}, \dots, \mathcal{J}_{q-1}\mathcal{O}_{V_1}) \ \text{ (total transform)}.$$

This is again a q-differential collection on V_1 .

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We also set

$$(\mathcal{GO}_V)_{I(H_1)^{qa}} = ((\mathcal{J}_1\mathcal{O}_{V_1}:I(H_1)^{qa}),\ldots,(\mathcal{J}_{q-1}\mathcal{O}_{V_1}:I(H_1)^{qa})),$$

which is also a q-differential collection on V_1 .



Let $V \xleftarrow{\pi} V_1 \supset H_1$ be a blow-up along a regular center $Z \subset V$.

Proposition (Fundamental point-wise inequality)

Let $\mathcal G$ be a q-differential collection, and assume that $\eta_x(\mathcal G)=qa+b$ for all $x\in \mathcal Z$, where $0\leq b< q$. Then

$$\eta_{\pi(x_1)}(\mathcal{G}) \ge \eta_{x_1}(\mathcal{G}_1), \quad \forall x_1 \in V_1$$

for
$$\mathcal{G}_1 := (\mathcal{GO}_{V_1})_{I(H_1)^{q_a}}$$
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for $\mathcal{G}_1 := (\mathcal{GO}_{V_1})_{I(H_1)^{q_a}}$.

In the case $\mathcal{G}=\mathcal{G}(\mathcal{M})$ for an \mathcal{O}_V^q -module $\mathcal{M}\subset\mathcal{O}_V$, the *a*-transform $\mathcal{M}_a\subset\mathcal{O}_{V_1}$ is then defined, and we have CLAIM:

$$\mathcal{G}(\mathcal{M}_a) \subset \mathcal{G}_1$$

hence $\eta_{x_1}(\mathcal{M}_a) \geq \eta_{x_1}(\mathcal{G}_1)$ for all $x_1 \in V_1$. Therefore, we cannot deduce the point-wise inequality $\eta_{\pi(x_1)}(\mathcal{M}) \geq \eta_{x_1}(\mathcal{M}_a)$.

For a ring R and a collection of ideals $\Lambda = \{I_1, \ldots, I_r\}$ we define

$$\mathsf{Diff}^i_{R,\Lambda,+} := \mathsf{Diff}^i_{R,\mathit{I}_1,+} \cap \dots \cap \mathsf{Diff}^i_{R,\mathit{I}_r,+}$$

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For an *F*-finite regular scheme V, a collection of ideals $\Lambda = \{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ on V, and an \mathcal{O}_V^q -module $\mathcal{M} \subset \mathcal{O}_V$, there coherent ideals

$$\mathit{Diff}^1_{V,\Lambda,+}(\mathscr{M})\subseteq \mathit{Diff}^2_{V,\Lambda,+}\subseteq\cdots\subseteq \mathit{Diff}^{q-1}_{V,\Lambda,+}(\mathscr{M})$$

s. t. for all $x \in V$ we have

$$(Diff^{i}_{V,\Lambda,+}(\mathscr{M}))_{x} = Diff^{i}_{\mathscr{O}_{V,x},\Lambda_{x},+}(\mathscr{M}_{x}), \quad \forall x \in V.$$

For a ring R and a collection of ideals $\Lambda = \{I_1, \dots, I_r\}$ we define

$$\mathsf{Diff}^i_{R,\Lambda,+} := \mathsf{Diff}^i_{R,I_1,+} \cap \cdots \cap \mathsf{Diff}^i_{R,I_r,+}$$

For an F-finite regular scheme V, a collection of ideals $\Lambda = \{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ on V, and an \mathcal{O}_{V}^{q} -module $\mathscr{M}\subset\mathcal{O}_{V}$, there coherent ideals

$$Diff^1_{V,\Lambda,+}(\mathscr{M}) \subseteq Diff^2_{V,\Lambda,+} \subseteq \cdots \subseteq Diff^{q-1}_{V,\Lambda,+}(\mathscr{M})$$

s. t. for all $x \in V$ we have

$$(Diff^{i}_{V,\Lambda,+}(\mathscr{M}))_{x} = Diff^{i}_{\mathcal{O}_{V,x},\Lambda_{x},+}(\mathscr{M}_{x}), \quad \forall x \in V.$$

Proposition

If \mathcal{L} is any invertible ideal included in $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_r$, then

$$\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) := ((Diff^1_{V,\Lambda,+}(\mathcal{M}) : \mathcal{L}^1), \dots, (Diff^{q-1}_{V,\Lambda,+}(\mathcal{M}) : \mathcal{L}^{q-1}))$$

is a q-differential collection. In addition, $\mathcal{G}(\mathcal{M}) \subseteq \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})$

Day 5: Multiplicity along points of a radicial covering of a regular variety

Assume that $\Lambda = \{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ is a collection of ideals of regular hypersurfaces with normal crossings.

Let $V \xleftarrow{\pi} V_1 \supset H_1$ be the blow-up along a regular center Z having normal crossings with these hypersurfaces.

Let $\mathcal{I}'_1,\ldots,\mathcal{I}'_r$ denote the strict transforms of $\mathcal{I}_1,\ldots,\mathcal{I}_r$.

Then $\Lambda_1:=\{\mathcal{I}_1',\ldots,\mathcal{I}_r',\mathcal{I}(H_1)\}$ has also normal crossings.

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Lemma (Giraud's lemma)

For any \mathcal{O}_{V}^{q} -module \mathcal{M} ,

$$\mathcal{I}(\textit{H}_1)^i(\textit{Diff}^i_{V,\Lambda,+}(\mathscr{M})\mathcal{O}_{V_1})\subseteq \textit{Diff}^i_{V_1,\Lambda_1,+}(\mathscr{M}\mathcal{O}^q_{V_1}),\ i=1,\dots,q-1.$$

Consider 3-uplas $(\mathcal{M}, \Lambda, \mathcal{L})$

 $\mathscr{M}\subset \mathcal{O}_V$ an \mathcal{O}_V^q -module

 $\Lambda = \{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ ideals of regular hypersurfaces with normal crossings

 $\mathcal{L} \subseteq \mathcal{I}_1 \cap \dots \cap \mathcal{I}_r$ an invertible ideal

Consider 3-uplas $(\mathcal{M}, \Lambda, \mathcal{L})$

 $\mathcal{M}\subset\mathcal{O}_V$ an \mathcal{O}_V^q -module $\Lambda=\{\mathcal{I}_1,\ldots,\mathcal{I}_r\}$ ideals of regular hypersurfaces with normal crossings $\mathcal{L}\subseteq\mathcal{I}_1\cap\cdots\cap\mathcal{I}_r$ an invertible ideal

We associate to this 3-upla the q-differential collection.

$$\leadsto \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) := ((Diff^1_{V,\Lambda,+}(\mathcal{M}) : \mathcal{L}^1), \dots, (Diff^{q-1}_{V,\Lambda,+}(\mathcal{M}) : \mathcal{L}^{q-1})) \supset \mathcal{G}(\mathcal{M})$$

and set

$$aq + b := \max\{\eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) : x \in V\}, \ 0 \le b < q.$$

We choose a regular center $Z \subset V$ having normal crossings with the hypersurfaces and perform the blow-up $V \xleftarrow{\pi} V_1 \supset H_1$ along Z.

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 $\mathcal{M}\subset\mathcal{O}_V$ an \mathcal{O}_V^q -module $\Lambda=\{\mathcal{I}_1,\ldots,\mathcal{I}_r\}$ ideals of regular hypersurfaces with normal crossings $\mathcal{L}\subseteq\mathcal{I}_1\cap\cdots\cap\mathcal{I}_r$ an invertible ideal

We associate to this 3-upla the q-differential collection.

$$\rightsquigarrow \mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L}) := ((\mathsf{Diff}^1_{V, \Lambda, +}(\mathcal{M}) : \mathcal{L}^1), \dots, (\mathsf{Diff}^{q-1}_{V, \Lambda, +}(\mathcal{M}) : \mathcal{L}^{q-1})) \supset \mathcal{G}(\mathcal{M})$$

and set

$$aq + b := \max\{\eta_x(\mathcal{G}(\mathcal{M}, \Lambda, \mathcal{L})) : x \in V\}, \ 0 \le b < q.$$

We choose a regular center $Z \subset V$ having normal crossings with the hypersurfaces and perform the blow-up $V \xleftarrow{\pi} V_1 \supset H_1$ along Z.

The a-transform of $(\mathcal{M}, \Lambda, \mathcal{L})$ is the 3-tuple $(\mathcal{M}_1, \Lambda_1, \mathcal{L}_1)$ on V_1 defined as:

 \mathcal{M}_1 the a-transform of \mathcal{M} $\Lambda_1 := \{\mathcal{I}'_1, \dots, \mathcal{I}'_r, I(H_1)\}, \ \mathcal{I}'_i$ the strict transform of \mathcal{I}_i $\mathcal{L}_1 = (\mathcal{LO}_{V_1})\mathcal{I}(H_1)$

$$\rightsquigarrow \mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) := ((\mathit{Diff}^1_{V_1, \Lambda_1, +}(\mathscr{M}_1) : \mathcal{L}^1_1), \ldots, (\mathit{Diff}^{q-1}_{V_1, \Lambda_1, +}(\mathscr{M}_1) : \mathcal{L}^{q-1}_1))$$

Theorem (Fundamental point-wise inequality)

There is an inclusion of q-differential collections

$$\begin{split} &(\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L})\mathcal{O}_{V_1})_{I(H_1)^{q_2}} \subseteq \mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) \\ \Rightarrow & \eta_{x_1}(\mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1)) \leq \eta_{\pi(x_1)}\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L}), \quad \forall x_1 \in V_1 \end{split}$$

In applications:

$$V_0 \stackrel{\pi_1}{\longleftarrow} V_1 \stackrel{\pi_2}{\longleftarrow} \dots \stackrel{\pi_r}{\longleftarrow} V_r$$

$$(\mathcal{M}_0, \Lambda_0, \mathcal{L}_0) \qquad (\mathcal{M}_1, \Lambda_1, \mathcal{L}_1) \qquad \dots \qquad (\mathcal{M}_r, \Lambda_r, \mathcal{L}_r)$$

where $\mathcal{M}_0 := \mathcal{M}$, $\Lambda_0 =$, and $\mathcal{L}_0 = \mathcal{O}_V$.

We get
$$\eta(\mathscr{M}_0, \Lambda_0, \mathcal{L}_0) = \eta(\mathscr{M})$$
 and

$$\max \eta(\mathscr{M}) = \max \eta(\mathscr{M}_0, \Lambda_0, \mathcal{L}_0) \ge \max \eta(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) \ge \cdots \ge \max \eta(\mathscr{M}_r, \Lambda_r, \mathcal{L}_r)$$

Thanks!