Day 5: Multiplicity along points of a radicial covering of a regular variety

D. Sulca and O. Villamayor

June 18, 2019

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δ is called transversal if $F_d(X) \neq \emptyset$

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δ_1 finite of generic rank d.

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which in turn can be read off from

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where $J\mathcal{O}_{V_1} = I(H_1)^b J_1$, with $H_1 \subset V_1$ the exceptional hypersurface.

Roll in resolution of singularities

Goal: Define a sequence of blow-ups along regular centers included in the set of d-fold points

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_r$$

with

$$d = \max \operatorname{mult}_X = \max \operatorname{mult}_{X_1} = \cdots = \max \operatorname{mult}_{X_{r-1}} > \max \operatorname{mult}_{X_r}.$$

This is equivalent to defining a sequence of blow-ups

$$V \longleftarrow V_1 \longleftarrow V_2 \qquad \cdots \longleftarrow V_r$$

where we set $(J_0, b) := (J, b)$ as before, and for $i > 0$,

- $V_{i-1} \leftarrow V_i \supset H_i$ blow-up along a regular $Z_{i-1} \subset \operatorname{Sing}(J_{i-1}, b)$
- $J_{i-1}\mathcal{O}_{V_i} = I(H_i)^b J_i$
- $\operatorname{Sing}(J_r, b) = \emptyset$.

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Summarizing: Radicial morphisms $X \to V$ are viewed from $\mathcal{O}_V^q$-modules $\mathscr{M} \subset \mathcal{O}_V$.

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Corollary

For an $\mathcal{O}_V^q$-module $\mathscr{M} \subset \mathcal{O}_V$ with associated morphism $\delta : X \to V$ we have

$$\delta(F_d(X)) = \{x \in V : \mathscr{M}_x \subseteq \mathcal{O}_{V,x}^q + m_{V,x}^q\}$$

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R a ring of characteristic p.
A subset $\{z_1, \dots, z_n\}$ is a (finite) p-basis if R is a free R^p-module with basis

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- $(R, \mathfrak{m})$ local, regular, F-finite, has a p-basis of the form

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$\Rightarrow$ R is free R^q-module with basis $\{\mathbf{x}^\alpha \mathbf{y}^\beta : (\alpha, \beta) \in \mathcal{A}_q\}$.

$$\forall f \in R,\ f = \sum_{\mathcal{A}_q} c^q_{\alpha,\beta} \mathbf{x}^\alpha \mathbf{y}^\beta \text{ (q-expansion)},$$
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Lemma

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Corollary

1) If $\mathfrak{p} \subset R$ is a regular prime, then $R^q + \mathfrak{p}^n = R \cap (R^q_{\mathfrak{p}} + \mathfrak{p}^n R_{\mathfrak{p}})$.

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1) If $\mathfrak{p} \subset R$ *is a regular prime, then* $R^q + \mathfrak{p}^n = R \cap (R^q_{\mathfrak{p}} + \mathfrak{p}^n R_{\mathfrak{p}})$.
2) For an $\mathcal{O}^q_V$*-module* $\mathscr{M} \subset \mathcal{O}_V$ *and* $Z \subset V$ *regular, t.f.a.e.:*

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$$D_{\gamma,\delta}(f) = \sum_{\mathcal{A}_q} c_{\alpha,\beta}^q \binom{\alpha}{\gamma}\binom{\beta}{\delta}\mathbf{x}^{\alpha-\gamma}\mathbf{y}^{\beta-\delta}$$

$$\mathbf{x}^\gamma D_{\gamma,\delta}(f) = \sum_{\mathcal{A}_q} c_{\alpha,\beta}^q \binom{\alpha}{\gamma}\binom{\beta}{\delta}\mathbf{x}^{\alpha}\mathbf{y}^{\beta-\delta}$$

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5. $f \in R^q + \mathfrak{m}^{qa}$ if and only if $\mathrm{Diff}_{R,+}^{q-1}(f) \subseteq \mathfrak{m}^{q(a-1)+1}$.
   In particular, $f \in R^q + \mathfrak{m}^q$ if and only if $\mathrm{Diff}_{R,+}^{q-1}(f) \subseteq \mathfrak{m}$.

Moving to the global setting

Fix an $\mathcal{O}_V^q$-module $\mathscr{M} \subset \mathcal{O}_V$ and an ideal $\mathcal{J} \subset \mathcal{O}_V$.

Then there are coherent ideals on V

$$Diff_V^0(\mathcal{J}) \subset Diff_V^1(\mathcal{J}) \subset Diff_V^2(\mathcal{J}) \subset \cdots$$
$$Diff_{V,+}^1(\mathscr{M}) \subset \cdots \subset Diff_{V,+}^{q-1}(\mathscr{M})$$

s. t. for all $x \in V$,

$$(Diff_V^k(\mathcal{J}))_x = \mathrm{Diff}_{\mathcal{O}_{V,x}}^k(\mathcal{J}_x)$$
$$(Diff_{V,+}^k(\mathscr{M}))_x = \mathrm{Diff}_{\mathcal{O}_{V,x},+}^k(\mathscr{M}_x),\ k = 1, \ldots, q-1.$$

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$$\nu_x^{(q)}(\mathscr{M}) := \max\{n \in \mathbb{N} : \mathscr{M}_x \subset \mathcal{O}_{V,x}^q + m_{V,x}^n\} \text{ (order after cleaning)}$$
$$\eta_x(\mathscr{M}) := \min\{\nu_x(Diff_{V,+}^1(\mathscr{M})) + 1, \ldots, \nu_x(Diff_{V,+}^{q-1}(\mathscr{M})) + q - 1\}$$

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Proposition

$\nu_x^{(q)}(\mathscr{M}) \leq \eta_x(\mathscr{M})$, and the equality holds if $q \nmid \nu_x^{(q)}(\mathscr{M})$. In addition,
$$\left\lfloor \frac{\eta_x(\mathscr{M})}{q} \right\rfloor = \left\lfloor \frac{\nu_x^{(q)}(\mathscr{M})}{q} \right\rfloor$$
In particular, $\{x \in V : \mathscr{M}_x \subseteq \mathcal{O}_{V,x}^q + m_{V,x}^{qa}\} = \{x \in V : \eta_x(\mathscr{M}) \geq qa\}$.

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Lemma

For a regular hypersurface $H \subset V$,

$$\nu_H(Diff^{q-1}_{V,+}(\mathscr{M})) = qa$$

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Assume $a \geq 1$. Then

$$\mathcal{O}_V^q + \mathscr{M} = \mathcal{O}_V^q + (\mathcal{O}_V^q + \mathscr{M}) \cap \mathcal{I}(H)^{qa} = \mathcal{O}_V^q + (\mathcal{I}(H)^{(q)})^a \mathscr{M}_a$$

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Lemma

For $\delta : X \to V$ the morphism defined by $\mathscr{M}$, there are finite morphisms

$$\begin{array}{ccccccccc} & X & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \dots \longleftarrow & X_a \\ & \downarrow{\scriptstyle\delta} & & & & & & & \\ H \subset & V & & & & & & & \end{array}$$

that are blow-ups along regular centers $Y_i \subset F_d(X_i)$ whose image in V is H. In addition, the composition $X_a \to V$ corresponds to $\mathscr{M}_a$.

Let $V \leftarrow V_1 \supset H_1$ be a blow-up along a regular center Z.

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- For $i = 1, \ldots, q-1$, there are inclusions

$$\begin{aligned} I(H_1)^i (Diff^i_{V,+}(\mathscr{M})\mathcal{O}_{V_1}) &\subseteq Diff^i_{V_1,H_1,+}(\mathscr{M}\mathcal{O}^q_{V_1}) \\ &\subseteq Diff^i_{V_1,+}(\mathscr{M}\mathcal{O}^q_{V_1}) \subseteq Diff^i_{V,+}(\mathscr{M})\mathcal{O}_{V_1} \end{aligned}$$

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Lemma (Fundamental point-wise inequality)

If $\nu_x(\mathcal{J}) = b$ for all $x \in Z$, so that $\mathcal{J}\mathcal{O}_{V_1} = \mathcal{I}(H_1)^b\mathcal{J}_1$ for an ideal $\mathcal{J}_1$ on V_1, then

$$\nu_{\pi(x_1)}(\mathcal{J}) \geq \nu_{x_1}(\mathcal{J}_1), \quad \forall x_1 \in V_1.$$

Proposition

Let $V \overset{\pi}{\longleftarrow} V_1 \supset H_1$ be a blow-up along a regular center $Z \subset V$, and set

$$\eta_Z(\mathscr{M}) = qa + b, \quad \text{with} \quad 0 \leq b < q, \text{ and assume } a \geq 1.$$

Then

$$\nu_{H_1}(Diff^{q-1}_{V_1,+}(\mathscr{M}\mathcal{O}^q_{V_1})) = qa,$$

so that for an $\mathcal{O}^q_{V_1}$-module $\mathscr{M}_a$, we have

$$\mathcal{O}^q_{V_1} + \mathscr{M}\mathcal{O}^q_{V_1} = \mathcal{O}^q_{V_1} + (\mathcal{I}(H_1)^{(q)})^a \mathscr{M}_a$$

In addition, there exists a commutative diagram

$$\begin{array}{ccccccccc}
X & \xleftarrow{\pi_\delta} & (X \times_V V_1)_{red} & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_a} & X_a \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_\pi} & & \downarrow{\scriptstyle \delta_1} & & & & \downarrow{\scriptstyle \delta_a} \\
V & \xleftarrow{\pi} & V_1 & \xleftarrow{=} & V_1 & \xleftarrow{=} & \cdots & \xleftarrow{=} & V_a \\
\mathscr{M} & & \mathscr{M}\mathcal{O}^q_{V_1} & & \mathscr{M}_1 & & & & \mathscr{M}_a
\end{array}$$

with $\pi_\delta f_1$ the blow-up of X along the center $Y \subset F_d(X)$ s.t. $Z = \delta(Y)$.

q-differential collection

A q-**differential collection** on V is a sequence of ideals

$$\mathcal{G} = (\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_{q-1})$$

such that $Diff^i_V(\mathcal{J}_j) \subseteq \mathcal{J}_{i+j}$.

Example: for an $\mathcal{O}_V^q$-module $\mathscr{M} \subset \mathcal{O}_V$,

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We associate with a q-differential collection the function

$$\eta_x(\mathcal{G}) = \min\{\nu_x(\mathcal{J}_1) + 1, \ldots, \nu_x(\mathcal{J}_{q-1}) + q - 1\}.$$

This defines an upper-semicontinuous function $V \to \mathbb{N}$.

Transformation of q-differential collections

Fix a blow-up $V \xleftarrow{\pi} V_1 \supset H_1$ along a regular center $Z \subset V$.

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Let $\mathcal{G} = (\mathcal{J}_1, \ldots, \mathcal{J}_{q-1})$ be a q-differential collection. We denote

$$\mathcal{G}\mathcal{O}_{V_1} = (\mathcal{J}_1\mathcal{O}_{V_1}, \ldots, \mathcal{J}_{q-1}\mathcal{O}_{V_1}) \ \text{(total transform)}.$$

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We also set

$$(\mathcal{G}\mathcal{O}_V)_{I(H_1)^{qa}} = ((\mathcal{J}_1\mathcal{O}_{V_1} : I(H_1)^{qa}), \ldots, (\mathcal{J}_{q-1}\mathcal{O}_{V_1} : I(H_1)^{qa})),$$

which is also a q-differential collection on V_1.

Let $V \overset{\pi}{\longleftarrow} V_1 \supset H_1$ be a blow-up along a regular center $Z \subset V$.

Proposition (Fundamental point-wise inequality)

Let $\mathcal{G}$ be a q-differential collection, and assume that $\eta_x(\mathcal{G}) = qa + b$ for all $x \in Z$, where $0 \leq b < q$. Then

$$\eta_{\pi(x_1)}(\mathcal{G}) \geq \eta_{x_1}(\mathcal{G}_1), \quad \forall x_1 \in V_1$$

for $\mathcal{G}_1 := (\mathcal{G}\mathcal{O}_{V_1})_{I(H_1)^{qa}}$.

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In the case $\mathcal{G} = \mathcal{G}(\mathscr{M})$ for an $\mathcal{O}_V^q$-module $\mathscr{M} \subset \mathcal{O}_V$, the a-transform $\mathscr{M}_a \subset \mathcal{O}_{V_1}$ is then defined, and we have CLAIM:

$$\mathcal{G}(\mathscr{M}_a) \subseteq \mathcal{G}_1$$

hence $\eta_{x_1}(\mathscr{M}_a) \geq \eta_{x_1}(\mathcal{G}_1)$ for all $x_1 \in V_1$. Therefore, **we cannot deduce the point-wise inequality** $\eta_{\pi(x_1)}(\mathscr{M}) \geq \eta_{x_1}(\mathscr{M}_a)$.

Constructions using logarithmic differential operators

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For a ring R and a collection of ideals $\Lambda = \{I_1, \dots, I_r\}$ we define

$$\mathrm{Diff}^i_{R,\Lambda,+} := \mathrm{Diff}^i_{R,I_1,+} \cap \cdots \cap \mathrm{Diff}^i_{R,I_r,+}$$

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For an F-finite regular scheme V, a collection of ideals $\Lambda = \{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$ on V, and an $\mathcal{O}_V^q$-module $\mathscr{M} \subset \mathcal{O}_V$, there coherent ideals

$$\mathit{Diff}^1_{V,\Lambda,+}(\mathscr{M}) \subseteq \mathit{Diff}^2_{V,\Lambda,+} \subseteq \cdots \subseteq \mathit{Diff}^{q-1}_{V,\Lambda,+}(\mathscr{M})$$

s. t. for all $x \in V$ we have

$$(\mathit{Diff}^i_{V,\Lambda,+}(\mathscr{M}))_x = \mathrm{Diff}^i_{\mathcal{O}_{V,x},\Lambda_x,+}(\mathscr{M}_x), \quad \forall x \in V.$$

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Proposition

If $\mathcal{L}$ is any invertible ideal included in $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_r$, then

$$\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L}) := ((Diff^1_{V,\Lambda,+}(\mathscr{M}) : \mathcal{L}^1), \ldots, (Diff^{q-1}_{V,\Lambda,+}(\mathscr{M}) : \mathcal{L}^{q-1}))$$

is a q-differential collection. In addition, $\mathcal{G}(\mathscr{M}) \subseteq \mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L})$

Assume that $\Lambda = \{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$ is a collection of ideals of regular hypersurfaces with normal crossings.

Let $V \stackrel{\pi}{\longleftarrow} V_1 \supset H_1$ be the blow-up along a regular center Z having normal crossings with these hypersurfaces.

Let $\mathcal{I}'_1, \ldots, \mathcal{I}'_r$ denote the strict transforms of $\mathcal{I}_1, \ldots, \mathcal{I}_r$.

Then $\Lambda_1 := \{\mathcal{I}'_1, \ldots, \mathcal{I}'_r, \mathcal{I}(H_1)\}$ has also normal crossings.

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Lemma (Giraud's lemma)

For any $\mathcal{O}_V^q$-module $\mathscr{M}$,

$$\mathcal{I}(H_1)^i (Diff^{\,i}_{V,\Lambda,+}(\mathscr{M})\mathcal{O}_{V_1}) \subseteq Diff^{\,i}_{V_1,\Lambda_1,+}(\mathscr{M}\mathcal{O}^q_{V_1}), \;\; i = 1, \dots, q-1.$$

Consider 3-uplas $(\mathscr{M}, \Lambda, \mathcal{L})$

$\mathscr{M} \subset \mathcal{O}_V$ an $\mathcal{O}_V^q$-module
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We associate to this 3-upla the q-differential collection.

$$\rightsquigarrow \mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L}) := ((Diff^1_{V,\Lambda,+}(\mathscr{M}) : \mathcal{L}^1), \ldots, (Diff^{q-1}_{V,\Lambda,+}(\mathscr{M}) : \mathcal{L}^{q-1})) \supset \mathcal{G}(\mathscr{M})$$

and set

$$aq + b := \max\{\eta_x(\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L})) : x \in V\}, \quad 0 \leq b < q.$$

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The *a-transform of* $(\mathscr{M}, \Lambda, \mathcal{L})$ is the 3-tuple $(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1)$ on V_1 defined as:

$\mathscr{M}_1$ the a-transform of $\mathscr{M}$
$\Lambda_1 := \{\mathcal{I}'_1, \dots, \mathcal{I}'_r, I(H_1)\}$, $\mathcal{I}'_i$ the strict transform of $\mathcal{I}_i$
$\mathcal{L}_1 = (\mathcal{L}\mathcal{O}_{V_1})\mathcal{I}(H_1)$

$$\rightsquigarrow \mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) := ((Diff^1_{V_1,\Lambda_1,+}(\mathscr{M}_1) : \mathcal{L}_1^1), \dots, (Diff^{q-1}_{V_1,\Lambda_1,+}(\mathscr{M}_1) : \mathcal{L}_1^{q-1}))$$

Theorem (Fundamental point-wise inequality)

There is an inclusion of q-differential collections

$$(\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L})\mathcal{O}_{V_1})_{I(H_1)^{qa}} \subseteq \mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1)$$

$$\Rightarrow \quad \eta_{x_1}(\mathcal{G}(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1)) \leq \eta_{\pi(x_1)}\mathcal{G}(\mathscr{M}, \Lambda, \mathcal{L}), \quad \forall x_1 \in V_1$$

In applications:

$$\begin{array}{ccccccc} V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r \\ (\mathscr{M}_0, \Lambda_0, \mathcal{L}_0) & & (\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) & & \dots & & (\mathscr{M}_r, \Lambda_r, \mathcal{L}_r) \end{array}$$

where $\mathscr{M}_0 := \mathscr{M}$, $\Lambda_0 =$, and $\mathcal{L}_0 = \mathcal{O}_V$.

We get $\eta(\mathscr{M}_0, \Lambda_0, \mathcal{L}_0) = \eta(\mathscr{M})$ and

$$\max \eta(\mathscr{M}) = \max \eta(\mathscr{M}_0, \Lambda_0, \mathcal{L}_0) \geq \max \eta(\mathscr{M}_1, \Lambda_1, \mathcal{L}_1) \geq \cdots \geq \max \eta(\mathscr{M}_r, \Lambda_r, \mathcal{L}_r)$$

Thanks!