Background independent exact renormalisation

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Background dependence

- General relativity is a theory without a fixed background geometry
- Continuum approaches to quantum gravity utilise a background metric

$$\Gamma = \Gamma[g_{\mu\nu}, \bar{g}_{\mu\nu}]$$

- This typically breaks diffeomorphism invariance

Background dependence

- A background metric is typically introduced to a) fix the gauge b) To regularise the theory
- Consequence: Infinite avatars of Newton's constant.
- Couplings are related by modified Ward identities.
- Can we have a single metric effective action?

Background independence

- Here I will discuss an approach to quantum gravity that is manifestly background independent and works without fixing the gauge.
- Can be applied to: Gravity, Gauge theory, ...

Background independent exact renormalisation

- Regularise the path integral.
- Derive an exact RG flow equation for the Wilsonian effective action.

Regularisation

- First step is to regularise the path integral in a diffeomorphism invariant manner.
- Regulator: Non-local action + Pauli-Villars determinants (a version of Slavnov's gauge invariant regularisation '74)

Regularisation

- QFT requires a UV regulator

$$P(p^2) = \frac{1}{p^2} \to \frac{1}{p^2} e^{-p^2/\Lambda^2}$$

- This can be achieved by modifying the action to depend explicitly on the cutoff scale

$$\frac{1}{2}\int_x \partial_\mu\phi\partial^\mu\phi \to \frac{1}{2}\int_x \partial_\mu\phi\, e^{-\partial^2/\Lambda^2}\,\partial^\mu\phi$$

Regularisation

- For gravity we can regularise the propagator by adding non-local terms to the Einstein Hilbert action

$$-\int d^x \sqrt{g} R \to \int \sqrt{g}\left[-R + G^{\mu\nu}\frac{1}{-\nabla^2}(e^{-\nabla^2/\Lambda^2}-1)R_{\mu\nu}\right]$$

- Action: regularised part plus an `interaction part'

$$S = \hat{S}_\Lambda + \mathcal{S}$$

Regularisation

- Modifying the action with infinite order derivatives only regulates multi-loop diagrams
- One loop diagrams are still divergent (Slavnov '73)
- Simple example: A scalar field in curved spacetime

$$\frac{1}{2}\mathrm{Tr}\log(-\nabla^2) \to \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 \mathrm{e}^{-\nabla^2/\Lambda^2})$$

Regularisation

- To regularise the path integral we must also modify the measure to include Pauli-Villars determinants e.g.

$$\int d\phi \to \int d\phi \sqrt{\det\left[-\nabla^2 e^{-\nabla^2/\Lambda^2} + \Lambda^2\right]}$$

- Regularised one-loop trace e.g.

$$\frac{1}{2}\mathrm{Tr}\log(-\nabla^2) \to \frac{1}{2}\mathrm{Tr}\log\left(\frac{-\nabla^2 \mathrm{e}^{-\nabla^2/\Lambda^2}}{-\nabla^2 \mathrm{e}^{-\nabla^2/\Lambda^2} + \Lambda^2}\right)$$

Regularisation

- Regularised integration over geometries

$$\int \prod_{x,\mu\nu} dg_{\mu\nu}(x) \frac{\sqrt{\det G_\Lambda[g]}}{\int \prod_{x,\mu} d\xi^\mu(x) \sqrt{\det H_\Lambda}[g]}$$

- The metrics of the space of spacetime metrics and the space of diffeomorphisms involve Pauli-Villars determinants.

- Choices of Pauli-Villars determinants can fully regularise the path integral

Exact renormalisation group

- The exact renormalisation group is a method that ensures that physics is invariant under a change in the cutoff scale.
- The couplings of the theory should depend on the cutoff scale

$$\lambda \to \lambda(\Lambda) \ \text{ such that } \ \frac{\partial}{\partial \Lambda} \mathcal{Z} = 0$$

- Observables extracted from the path integral independent of the cutoff

Exact renormalisation group

- A change in the cutoff is accompanied by a field transformation

$$-\int d\phi\,\Lambda\frac{\partial}{\partial\Lambda}e^{-S_\Lambda[\phi]} = \int d\phi\int_x \frac{\delta}{\delta\phi^x}\left(\Psi^x[\phi]e^{-S_\Lambda[\phi]}\right) = 0$$

- Where we can understand the field transformation as an averaging of the bare fields

$$e^{-S_\Lambda[\phi]} = \int d\varphi\,\delta[\phi - b_\Lambda[\varphi]]e^{-S_{\Lambda_0}[\varphi]}$$

- Then the averaging vector is given by

$$\Psi[b_\Lambda] = \Lambda\partial_\Lambda b_\Lambda[\varphi]$$

Exact renormalisation group

- The ERGE or flow equation for the Wilsonian action is given by

$$\Lambda\partial_\Lambda S + \int_x \Psi^x \frac{\partial S}{\partial \phi^x} = -\int_x \frac{\partial \Psi^x}{\partial \phi^x}$$

- To have a well defined flow we should demand that:

$$\Psi^x = -\frac{1}{2}\int_y \mathcal{K}^{xy}\frac{\delta \mathcal{S}}{\delta \phi^y} + \psi^x$$

- The flow then has the form of a heat equation

$$(\mathcal{F} + \Lambda\partial_\Lambda)e^{-\mathcal{S}} = 0$$

Exact renormalisation group

- Requirements for the averaging vector:

1. Solutions to the ERGE should lead to regulated path integrals.

2. The flow should be quasi-local meaning that it has a well behaved derivative expansion.

Flow as a diffeomorphism on the space of geometries

- Geometric flow:

$$\Lambda D_\Lambda \, \tilde{\mu}[g] e^{-S[g]} = 0$$

$$\tilde{\mu}[g] = \frac{\sqrt{\det G_\Lambda[g]}}{\sqrt{\det H_\Lambda[g]}} \qquad\qquad \Lambda D_\Lambda \equiv \Lambda \partial_\Lambda + \mathcal{L}_\Psi$$

$$\Lambda D_\Lambda S = \frac{1}{2} \mathrm{Tr}[G^{-1} \Lambda D_\Lambda G - H^{-1} \Lambda D_\Lambda H]$$

DeWitt Metric

- Since we work on the space of metrics it is convenient to utilise a DeWitt metric

$$\gamma_{\mu\nu\rho\sigma}(x,y) = \frac{\Lambda^2}{32\pi G_N}\frac{1}{2}\sqrt{g}\left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}\right)\delta(x,y)$$

- Indices are contracted using the DeWitt metric and functional derivatives are covariant with the Levi-Civita connection —> Field covariant

- Line element: $$ds^2 = \delta g \cdot \gamma \cdot \delta g$$

What doesn't work

- In analogy to regulating a scalar in curved spacetime one might try:

$$G = \hat{S}^{(2)} + \gamma$$

- This doesn't regularise the theory because the PV vertices diverge too fast for high momentum.

Two levels of regularisation

- Instead use that $\log(C^2) = 2\log(C)$

$$\hat{S} = \hat{s} + \frac{1}{2}\hat{s}^{(1)} \cdot \gamma^{-1} \cdot \hat{s}^{(1)}$$

- Regularised metric on geometries (same structure as Bakeyev,Slavnov '96)

$$G = (1 + \gamma^{-1} \cdot \hat{s}^{(2)}) \cdot \gamma \cdot (1 + \gamma^{-1} \cdot \hat{s}^{(2)})$$

- Extra longitudinal terms included in the Hessian which are regularised by the metric on the space of diffeos.

Two levels of regularisation

- The regularisation would be spoiled if $\mathcal{S}^{(n)} \sim \hat{S}^{(n)}$
- Regularisation is not spoilt for $\mathcal{S}^{(n)} \sim \hat{s}^{(n)}$
- Note: $\hat{S}^{(1)} = (1 + \gamma^{-1} \cdot \hat{s}^{(2)}) \cdot \hat{s}^{(1)}$
- UV cutoff in the flow equation: $C = \dfrac{1}{1 + \gamma^{-1} \cdot \hat{s}^{(2)}}$

‘classical flow’

- The lhs of the flow is given by

$$\Lambda\partial_\Lambda(\hat{s} + \frac{1}{2}\hat{s}^{(1)} \cdot \gamma^{-1} \cdot \hat{s}^{(1)} + \mathcal{S}) + (\hat{s}^{(1)} \cdot C^{-1} + \mathcal{S}^{(1)}) \cdot \psi - \frac{1}{2}(\hat{s}^{(1)} \cdot C^{-1} + \mathcal{S}^{(1)}) \cdot \mathcal{K} \cdot \mathcal{S}^{(1)} = \frac{1}{2}\mathrm{Tr}[...]$$

- Fix ψ by

$$\Lambda\partial_\Lambda \frac{1}{2}\hat{s}^{(1)} \cdot \gamma^{-1} \cdot \hat{s}^{(1)} + \hat{s}^{(1)} \cdot C^{-1} \cdot \psi = 0$$

$$\psi = -\frac{1}{2}C \cdot (\Lambda\partial_\Lambda \gamma^{-1} \cdot \hat{s}^{(1)} + \gamma^{-1} \cdot \Lambda\partial_\Lambda \hat{s}^{(1)})$$

- Take $\mathcal{K} = C \cdot \kappa$

The flow

- The lhs of the flow is given by

$$\Lambda\partial_\Lambda(\hat{s}+\mathcal{S})+\mathcal{S}^{(1)}\cdot\psi-\frac{1}{2}(\hat{s}^{(1)}+\mathcal{S}^{(1)}\cdot C)\cdot\kappa\cdot\mathcal{S}^{(1)}=\frac{1}{2}\mathrm{Tr}[...]$$

- One can further pick a convenient choice for κ

- The flow generates solutions which preserve the flow. One can show that the trace is UV finite as a consequence.

Conclusions and summary

- New diffeomorphism invariant flow equation
- No background, no gauge fixing, no ghosts
- First test: One-loop divergences of GR are reproduced
- Applications: Quantum gravity and gauge theories (e.g. QCD)