Three ways to solve critical φ^4 theory on 4-ε dimensional real projective space:

perturbation, bootstrap, and Schwinger-Dyson equation

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c.f. previous our work: *Mod. Phys. Lett.* A **32**, no. 07, 1750045 (2017) [arXiv:1611.06373 [hep-th]]

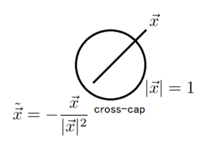
1. Introduction

We solve the two-point function of the lowest dimensional scalar operator in the critical ϕ^4 theory on $4-\epsilon$ dimensional real projective space in three different methods. The first is to use the conventional perturbation theory, and the second is to impose the crosscap bootstrap equation, and the third is to solve the Schwinger-Dyson equation under the assumption of conformal invariance. We find that the three methods lead to mutually consistent results but each has its own advantage.

As a theoretical interest, we also ask the following question: assuming that we know all the CFT data on a flat space-time, how much can we determine the CFT data on a curved space-time including a real projective space?

2. CFT on RP^d [Y. Nakayama 2016]

RP^d is defined by involution $\vec{x} \to -\frac{\vec{x}}{|\vec{x}|^2}$ on R^d



$$SO(d+1) \subset SO(d+1,1)$$

$$\langle O_i(\vec{x})\rangle^{\mathbb{RP}^d} = \frac{A_i^{\Omega}}{(1+|\vec{x}|^2)^{\Delta_i}}$$

$$\langle O_i(\vec{x}_1)O_j(\vec{x}_2)\rangle^{\mathbb{RP}^d} = \frac{(1+|\vec{x}_1|^2)^{\frac{-\Delta_i+\Delta_j}{2}}(1+|\vec{x}_2|^2)^{\frac{-\Delta_j+\Delta_i}{2}}}{|\vec{x}_1-\vec{x}_2|^{2\left(\frac{\Delta_i+\Delta_j}{2}\right)}} G_{ij}^{\Omega}(\eta)$$

$$\eta := \frac{|\vec{x}_1-\vec{x}_2|^2}{(1+|\vec{x}_1|^2)(1+|\vec{x}_2|^2)}$$: crosscap crossratio

crosscap bootstrap equation:

$$\boxed{\left(\frac{1-\eta}{\eta^2}\right)^{\frac{\Delta_i+\Delta_j}{6}} G_{ij}^{\Omega}(\eta) = \Omega_j{}^k \left(\frac{\eta}{(1-\eta)^2}\right)^{\frac{\Delta_i+\Delta_j}{6}} G_{ik}^{\Omega}(1-\eta)}$$

$$G_{ij}^{\Omega}(\eta) = \sum_k C_{ij}{}^k A_k^{\Omega} \eta^{\frac{\Delta_k}{2}} {}_2F_1\left(\frac{\Delta_i-\Delta_j+\Delta_k}{2}, \frac{\Delta_j-\Delta_i+\Delta_k}{2}; \Delta_k+1-\frac{d}{2}; \eta\right)$$

3. Perturbation c.f. bCFT$_d$ [McAvity-Osborn 1993]

$d = 4-\epsilon$ critical ϕ^4 theory

$$S[\phi,g] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4\right]$$

$$Z[g] = \int [\mathcal{D}\phi] \mathrm{e}^{-S[\phi,g]}$$

$$\langle\phi(\vec{x})\rangle_{\text{free}}^{\mathbb{RP}^d} = 0$$

$$\langle\phi^2(\vec{x})\rangle_{\text{free}}^{\mathbb{RP}^d} = \pm\frac{1}{4\pi^2}\frac{1}{(1+|\vec{x}|^2)^{2\Delta_\phi^{\text{free}}}}$$

$$\langle\phi(\vec{x})\phi(\vec{y})\rangle_{\text{free}}^{\mathbb{RP}^d} = \frac{1}{4\pi^2}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta_\phi^{\text{free}}}}\left[1 \pm \left(\frac{\eta}{1-\eta}\right)^{\Delta_\phi^{\text{free}}}\right]$$

$$\boxed{\begin{aligned}\langle\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d} &= \langle\phi(\vec{x})\phi(\vec{y})\rangle_{\text{free}}^{\mathbb{RP}^d} - \frac{g}{4!}\int \mathrm{d}^d z \langle\phi(\vec{x})\phi(\vec{y})\phi^4(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d} + O(g^2)\\ &= \langle\phi(\vec{x})\phi(\vec{y})\rangle_{\text{free}}^{\mathbb{RP}^d} - \frac{g}{4!}\int \mathrm{d}^d z \left[3\cdot\langle\phi(\vec{x})\phi(\vec{y})\rangle_{\text{free}}^{\mathbb{RP}^d}\left[\langle\phi^2(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\right]^2\right.\\ &\quad \left.+12\cdot\langle\phi(\vec{x})\phi(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\langle\phi(\vec{y})\phi(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\langle\phi^2(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\right] + O(g^2)\end{aligned}}$$

$$0 \le |z| \le 1$$

$$\begin{aligned}\left(\langle\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d}\right)^{(1)}\epsilon &:= -\frac{g}{4!}\int \mathrm{d}^d z \left[12\cdot\langle\phi(\vec{x})\phi(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\langle\phi(\vec{y})\phi(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\langle\phi^2(\vec{z})\rangle_{\text{free}}^{\mathbb{RP}^d}\right]\\ &= \frac{g}{2}\pi^2\left(\frac{1}{4\pi^2}\right)^3\left[\pm\frac{1}{1+2\vec{x}\cdot\vec{y}+|\vec{x}|^2|\vec{y}|^2}\ln\frac{|\vec{x}-\vec{y}|^2}{(1+|\vec{x}|^2)(1+|\vec{y}|^2)} + \frac{1}{|\vec{x}-\vec{y}|^2}\ln\frac{1+2\vec{x}\cdot\vec{y}+|\vec{x}|^2|\vec{y}|^2}{(1+|\vec{x}|^2)(1+|\vec{y}|^2)}\right]\end{aligned}$$

$$g_* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2) \qquad \gamma_{\phi^2} = (\gamma_{\phi^2})^{(1)}\epsilon + O(\epsilon^2) = \frac{1}{3}\epsilon + O(\epsilon^2)$$

results:

$$\boxed{\left(\langle\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d}\right)^{(1)}\epsilon = |\vec{x}-\vec{y}|^{-2}C_{\phi\phi}{}^I A_I^{\pm}\frac{(\gamma_{\phi^2})^{(1)}}{2}\epsilon\left[\pm\frac{\eta}{1-\eta}\ln\eta + \ln(1-\eta)\right]}$$

4. Bootstrap c.f. bCFT$_d$ [Liendo-Rastelli-Rees 2012]

crosscap bootstrap equation:

$$\boxed{G_{\phi\phi}^{\pm}(\eta) = \pm\left(\frac{\eta}{1-\eta}\right)^{\Delta_\phi} G_{\phi\phi}^{\pm}(1-\eta)}$$

$$\langle\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d} = |\vec{x}-\vec{y}|^{-2\Delta_\phi}G_{\phi\phi}^{\pm}(\eta)$$

$$[\phi]\times[\phi] = I + [\phi^2] + [\phi^4] + O(\epsilon^2)$$

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{2} + (\gamma_\phi)^{(1)}\epsilon + O(\epsilon^2)$$

$$\Delta_{\phi^2} = 2 - \epsilon + (\gamma_{\phi^2})^{(1)}\epsilon + O(\epsilon^2)$$

$$\Delta_{\phi^4} = 4 - 2\epsilon + (\gamma_{\phi^4})^{(1)}\epsilon + O(\epsilon^2)$$

$$C_{\phi\phi}{}^I A_I^{\pm} = \frac{1}{4\pi^2} : \text{normalization}$$

$$C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm} = (C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)} + (C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(1)}\epsilon + O(\epsilon^2)$$

$$C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm} = (C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}\epsilon + O(\epsilon^2)$$

$$\begin{aligned}&C_{\phi\phi}{}^I A_I^{\pm} + C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm}\eta^{\frac{\Delta_{\phi^2}}{2}}{}_2F_1\left(\frac{\Delta_{\phi^2}}{2},\frac{\Delta_{\phi^2}}{2};\Delta_{\phi^2}+1-\frac{d}{2};\eta\right)\\ &\quad + C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm}\eta^{\frac{\Delta_{\phi^4}}{2}}{}_2F_1\left(\frac{\Delta_{\phi^4}}{2},\frac{\Delta_{\phi^4}}{2};\Delta_{\phi^4}+1-\frac{d}{2};\eta\right)\\ &= \pm\left(\frac{\eta}{1-\eta}\right)^{\Delta_\phi}\left[C_{\phi\phi}{}^I A_I^{\pm} + C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm}(1-\eta)^{\frac{\Delta_{\phi^2}}{2}}{}_2F_1\left(\frac{\Delta_{\phi^2}}{2},\frac{\Delta_{\phi^2}}{2};\Delta_{\phi^2}+1-\frac{d}{2};1-\eta\right)\right.\\ &\quad \left.+C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm}(1-\eta)^{\frac{\Delta_{\phi^4}}{2}}{}_2F_1\left(\frac{\Delta_{\phi^4}}{2},\frac{\Delta_{\phi^4}}{2};\Delta_{\phi^4}+1-\frac{d}{2};1-\eta\right)\right]\end{aligned}$$

results:

$$\boxed{\begin{aligned}&(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)} = \pm C_{\phi\phi}{}^I A_I^{\pm}\\ &(\gamma_\phi)^{(1)} = 0\\ &(\gamma_{\phi^2})^{(1)} = \frac{4(C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}}{C_{\phi\phi}{}^I A_I^{\pm}}\\ &(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(1)} = -2(C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}\end{aligned}}$$

$$\boxed{G_{\phi\phi}^{\pm}(\eta) = C_{\phi\phi}{}^I A_I^{\pm}\left[1 \pm \left(\frac{\eta}{1-\eta}\right)^{1-\frac{\epsilon}{2}} + \frac{(\gamma_{\phi^2})^{(1)}}{2}\epsilon\left[\pm\frac{\eta}{1-\eta}\ln\eta + \ln(1-\eta)\right]\right] + O(\epsilon^2)}$$

5. Schwinger-Dyson equation

Schwinger-Dyson equation:

$$\boxed{\langle\Box_x\phi(\vec{x})\cdots\rangle^{\mathbb{RP}^d} = \langle\frac{g}{3!}\phi^3(\vec{x})\cdots\rangle^{\mathbb{RP}^d}}$$

c.f. CFT on R^d [Nii 2016]

Axiom I: The Wilson-Fisher fixed point has conformal symmetry.
Axiom II: If we take the $\epsilon \to 0$ limit, correlation functions in the interacting theory will approach the ones in the free theory.
Axiom III: From the Schwinger-Dyson equation, a particular primary operator in the free theory behaves as a descendant operator at the Wilson-Fisher fixed point. [Rychkov-Tan 2015]

$$\lim_{\epsilon\to 0}\langle\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d} = \langle\phi(\vec{x})\phi(\vec{y})\rangle_{\text{free}}^{\mathbb{RP}^d}$$

results:

$$\boxed{\begin{aligned}&C_{\phi\phi}{}^I A_I^{\pm} = \frac{1}{4\pi^2} : \text{normalization}\\ &(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)} = \pm\frac{1}{4\pi^2}(= \pm C_{\phi\phi}{}^I A_I^{\pm})\end{aligned}}$$

$$\langle\Box_x\phi(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d} = \frac{g}{3!}\langle\phi^3(\vec{x})\phi(\vec{y})\rangle^{\mathbb{RP}^d}$$

$$O(\epsilon\eta^0) : 4(\gamma_\phi)^{(1)}\epsilon|\vec{x}-\vec{y}|^{-4} = 0$$

$$\begin{aligned}O(\epsilon\eta^1) &: 2(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)}\left[(1-|\vec{x}|^2+|\vec{y}|^2)(\gamma_{\phi^2})^{(1)} - 2(\gamma_\phi)^{(1)}\right]\epsilon|\vec{x}-\vec{y}|^{-4}\eta\\ &= \pm\frac{g}{2}\left(\frac{1}{4\pi^2}\right)^2(1-|\vec{x}|^2+|\vec{y}|^2)|\vec{x}-\vec{y}|^{-4}\eta\end{aligned}$$

$$O(\epsilon\eta^2) : \left[-12(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)}(\gamma_\phi)^{(1)}\epsilon + 8(C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}\epsilon\right]|\vec{x}-\vec{y}|^{-4}\eta^2 = \frac{g}{2}\left(\frac{1}{4\pi^2}\right)^2|\vec{x}-\vec{y}|^{-4}\eta^2$$

$$(\gamma_\phi)^{(1)} = 0$$

$$(\gamma_{\phi^2})^{(1)}\epsilon = \frac{g}{16\pi^2}$$

$$(C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}\epsilon = \frac{g}{16}(C_{\phi\phi}{}^I A_I^{\pm})^2$$

$$\langle\Box_x\phi(\vec{x})\Box_y\phi(\vec{y})\rangle^{\mathbb{RP}^d} = \frac{g^2}{(3!)^2}\langle\phi^3(\vec{x})\phi^3(\vec{y})\rangle^{\mathbb{RP}^d}$$

$$O(\epsilon^2\eta^0) : C_{\phi\phi}{}^I A_I^{\pm}2^5(\gamma_\phi)^{(2)}\epsilon^2|\vec{x}-\vec{y}|^{-6} = \frac{g^2}{6}\left(\frac{1}{4\pi^2}\right)^3|\vec{x}-\vec{y}|^{-6}$$

$$O(\epsilon^2\eta) : 4(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)}(\gamma_{\phi^2})^{(1)}\left[1-(\gamma_{\phi^2})^{(1)}\right]\epsilon^2|\vec{x}-\vec{y}|^{-6}\eta = \pm\frac{g^2}{2}\left(\frac{1}{4\pi^2}\right)^3|\vec{x}-\vec{y}|^{-6}\eta$$

$$\begin{aligned}O(\epsilon^2\eta^2) &: \left[4(C_{\phi\phi}{}^{\phi^2}A_{\phi^2}^{\pm})^{(0)}(\gamma_{\phi^2})^{(1)}\left[1-(\gamma_{\phi^2})^{(1)}\right] + 16(C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm})^{(1)}\left[(\gamma_{\phi^4})^{(1)}-1\right]\right]\epsilon^2|\vec{x}-\vec{y}|^{-6}\eta^2\\ &= \begin{cases}\frac{5}{4}g^2\left(\frac{1}{4\pi^2}\right)^3|\vec{x}-\vec{y}|^{-6}\eta^2\\ \frac{1}{4}g^2\left(\frac{1}{4\pi^2}\right)^3|\vec{x}-\vec{y}|^{-6}\eta^2.\end{cases}\end{aligned}$$

$$(\gamma_\phi)^{(2)}\epsilon^2 = \frac{g^2}{3\cdot 4^3\cdot(4\pi^2)^2},$$

$$g_* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2),$$

$$(\gamma_{\phi^4})^{(1)}\epsilon = \begin{cases}6\cdot\frac{g}{16\pi^2}\\ 2\epsilon\end{cases}$$

results:

$$\boxed{\begin{aligned}&\gamma_\phi = \frac{1}{108}\epsilon^2 + O(\epsilon^3)\\ &\gamma_{\phi^2} = \frac{1}{3}\epsilon + O(\epsilon^2)\\ &\gamma_{\phi^4} = 2\epsilon + O(\epsilon^2)\\ &C_{\phi\phi}{}^{\phi^4}A_{\phi^4}^{\pm} = \frac{1}{4\cdot 3\cdot 4\pi^2}\epsilon + O(\epsilon^2)\end{aligned}}$$

6. Conclusions

The first method we used is the conventional perturbation theory. At order ϵ, the computation is straightforward and we can compute the CFT data with no difficulty. In particular, in the computation of the ϕ - ϕ two-point function, there is no necessity of the renormalization. Beyond this order, however, the computation becomes more involved and we have to perform the renormalization in the curved background. Note also that the $SO(d+1)$conformal symmetry on the $\mathbb{RP}^d$ is not manifest in this approach.

The second method we used is the crosscap bootstrap equation. This employs the conformal symmetry manifestly but does not specify the model. In general, we need the infinite number of primary operators to satisfy the crosscap bootstrap equation, but the special features of the CFT data of the critical ϕ^4 theory allow the truncation. We then found that the crosscap bootstrap equation possesses a one-parameter family of solutions. This corresponds to the existence of the critical $O(N)$ models that satisfy the same crosscap bootstrap equation. We found that once we fix this parameter, e.g. by specifying the anomalous dimension of the ϕ^2 operator, the solution is unique and coincide with the perturbative computation.

The third method we used is the Schwinger-Dyson equation combined with the conformal symmetry. This approach allows us to evaluate some of the CFT data at $O(\epsilon^2)$ without much a do about the renormalization. Furthermore, we can specify the coupling constant at the critical value even without computing the renormalization group beta function because we imposed the conformal symmetry. On the other hand, we find that not every CFT data is fixed in this approach. In other words, the two-point function as a solution of the Schwinger-Dyson equation contains an integration constant that we cannot determine from this approach alone.