A new large N expansion for tensor and matrix-tensor models

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Quantum and Gravity in Okinawa, OIST
25 July 2019

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2 Tensor models
- Preliminaries
- Interaction vertices and Feynman graphs
- Large N expansions

3 Matrix-tensor models
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Context

Random **matrix** models provide a description of random triangulated **surfaces**

- In perturbation theory, Feynman graphs correspond to ribbon graphs, dual to triangulated surfaces
- When N is large, the perturbative expansion can be reorganized as a series in powers of the small parameter $1/N$
- This $1/N$ series is governed by the **genus** g ('t Hooft - 1974)

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In the continuum limit, matrix models fall into the universality class of pure **two-dimensional quantum gravity**

Context

Original motivation for random **tensor** models → describe random geometries in **higher dimensions** (Ambjorn, Durhuus, Jonsson - 1991; Sasakura - 1991; Gross - 1992)

First $1/N$ expansions discovered almost 20 years later (Gurau - 2010; Bonzom, Gurau, Rivasseau - 2012)

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In this talk: **uncolored** tensor models based on a single real tensor of rank R

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- More recent motivation → connection with **SYK-like** physics and holography (Witten - 2016; Klebanov, Tarnopolsky - 2017)
- In this talk: **uncolored** tensor models based on a single real tensor of rank R
- **BGR scaling** → well-defined $1/N$ expansion governed by a new quantity called **Gurau degree** of the Feynman graphs
- Leading order graphs: degree zero - **melons**

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$$\rightarrow \quad T_{a_1 \dots a_R} = (X_{ab})_{\mu_1 \dots \mu_r} \quad \text{with} \quad r = R - 2$$

(Bonzom, Combes - 2014)

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  $\rightarrow$ New parameter D

$\longrightarrow$ **Matrix-tensor models** (Ferrari, Rivasseau, Valette - 2017)

Preliminaries

Basic variable: real rank-R tensor

$$T_{a_1 \dots a_R} \ , \quad a_i = 1, \dots, N \quad \text{for} \quad i = 1, \dots, R$$

transforming in the R-fundamental representation of the symmetry group $O(N)^R$

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Interested in $\mathrm{O}(N)^R$-invariant tensor models; i.e. models invariant under

$$T_{a_1 \dots a_R} \to T'_{a_1 \dots a_R} = O^{(1)}_{a_1 a'_1} O^{(2)}_{a_2 a'_2} \cdots O^{(R)}_{a_R a'_R} \ T_{a'_1 \dots a'_R}$$

Partition function - free energy

$$Z = \exp(-F) = \int [dT] \exp(-S(T))$$

General action

Interested in the combinatorics of the connected vacuum Feynman graphs

$\rightarrow$ **General action**

$$S = N^{R-1}\Big(T_{a_1 \dots a_R} T_{a_1 \dots a_R} + \sum_a \tau_a \, I_{\mathcal{B}_a}(T) \Big)$$

Invariant interaction terms

$$I_{\mathcal{B}_a}(T) \sim T_{a_1 \dots a_R} T_{b_1 \dots b_R} \cdots T_{c_1 \dots c_R}$$

where indices (of the same type) are contracted pairwise and summed over

Interaction vertices

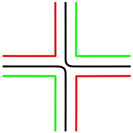
Two ways of representing the interaction vertices associated to interaction terms $I_{\mathcal{B}_a}$:

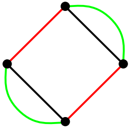
- **Stranded** representation: stranded graph vertices with R types of strands
- **Colored** representation: R-regular colored graphs, called *R-bubbles* and denoted by $\mathcal{B}_a$

Examples

Example with $R = 3$:

$$T_{a_1 a_2 a_3} T_{a_1 b_2 b_3} T_{b_1 b_2 b_3} T_{b_1 a_2 a_3} = T_{a_1 a_2 a_3} T_{a_1 b_2 b_3} T_{b_1 b_2 b_3} T_{b_1 a_2 a_3}$$





$$\deg \mathcal{B}_a = 0$$

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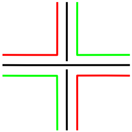
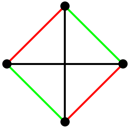




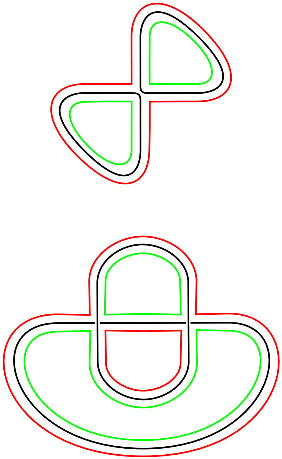
$$\deg \mathcal{B}_a = \frac{1}{2}$$

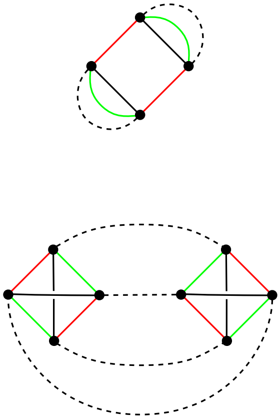
Feynman graphs

Also two representations:

- **Stranded** representation: interaction vertices connected by propagators composed of R strands
- **Colored** representation: $(R+1)$-bubbles, denoted by $\mathcal{B}$, with the additional color (color 0) connecting vertices of the interaction bubbles $\mathcal{B}_a$ in $\mathcal{B}$ (Wick contraction)

Examples





Large N expansion - BGR scaling

Large N **BGR scaling**:

$$\tau_a = N^{-\frac{2}{(R-2)!}\deg \mathcal{B}_a}\lambda_a$$

with λ_a fixed when $N \to \infty$

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$$\tau_a = N^{-\frac{2}{(R-2)!}\deg \mathcal{B}_a}\lambda_a$$

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Then, the free energy F admits a **well-defined** large N expansion

$$F = \sum_{L\in\mathbb{N}} F_L\, N^{R-L}$$

$\to$ large N expansion governed by the **Gurau degree**

$$L = \frac{2}{(R-1)!}\deg \mathcal{B} \in \mathbb{N}$$

$\to$ expansion parameter: $1/N$

Large N expansion - BGR scaling

Leading order graphs: graphs of degree zero - **melons**

Results:

- Melons can be constructed in a recursive way, they can be enumerated and they triangulate spheres in R dimensions
- In the continuum limit, they give rise to the universality class of branched polymers
- In quantum field theories in $d > 0$, they correspond to tadpoles

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$\rightarrow$ Interesting to **enlarge** the leading sector and try to generate richer emergent geometries or richer physics

Large N expansion - enhanced scaling

Large N **enhanced scaling**:

$$\tau_a = N^{+\frac{2}{(R-1)!}\deg \mathcal{B}_a} \mu_a$$

with μ_a fixed when $N \to \infty$

$\rightarrow$ enhancement for all non-melonic interactions ($\deg \mathcal{B}_a > 0$)

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Result: the free energy F still admits a **well-defined** large N expansion

$$F = \sum_{\ell \in \mathbb{N}} F_\ell \, N^{R - \frac{\ell}{R-1}}$$

$\rightarrow$ large N expansion governed by the **index**

$$\ell = 2 \, \mathrm{ind}_0 \, \mathcal{B} \in \mathbb{N}$$

$\rightarrow$ expansion parameter: $1/N^{\frac{1}{R-1}}$

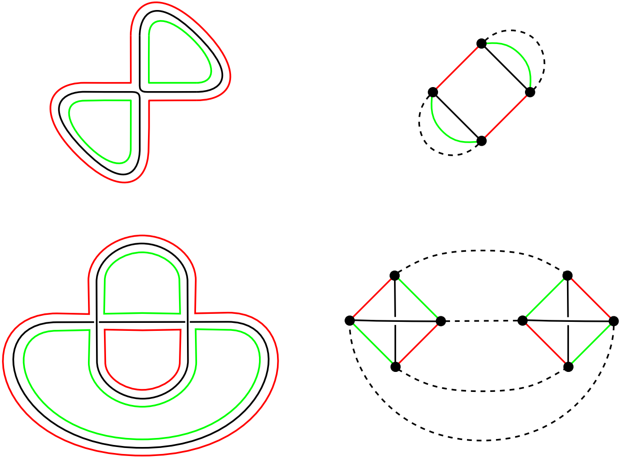
Large N expansion - enhanced scaling

Leading order graphs: graphs of index zero - **generalized melons**

Results:

- $\deg \mathcal{B} = 0 \Rightarrow \operatorname{ind} \mathcal{B} = 0$
- generalized melons form a strictly **larger** family of graphs than melons
- no general classification of generalized melons, only known for particular interactions
  $\rightarrow$ MST interactions, etc.
- critical behavior in the continuum limit not known in general
- enhanced scaling crucial for building tensor models with **SYK-like physics**

Melon versus generalized melon



Matrix-tensor models

Basic variables: real matrices X of size $N \times N$

$$(X_{ab})_{\mu_1 \dots \mu_r} = X_{ab\mu_1 \dots \mu_r} \ , \quad 1 \leq a, b \leq N \ , \quad 1 \leq \mu_i \leq D$$

$\rightarrow$ set of D^r matrices / tensor with $r+2$ indices

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$\rightarrow$ set of D^r matrices / tensor with $r+2$ indices

Goal: to construct $\mathrm{O}(N)^2 \times \mathrm{O}(D)^r$-invariant matrix-tensor models; i.e. models invariant under

$$(X_{ab})_{\mu_1 \ldots \mu_r} \rightarrow (X'_{ab})_{\mu_1 \ldots \mu_r} = O^{(1)}_{aa'} O^{(2)}_{bb'} O^{(3)}_{\mu_1 \mu'_1} \cdots O^{(r+2)}_{\mu_r \mu'_r} \ (X_{a'b'})_{\mu'_1 \ldots \mu'_r}$$

General action

$$S = ND^r \Big(\operatorname{tr} X_{\mu_1 \ldots \mu_r} X^{\mathrm{T}}_{\mu_1 \ldots \mu_r} + \sum_a \tau_a \, I_{\mathcal{B}_a}(X) \Big)$$

Large N and large D expansions - enhanced scaling

Result: the free energy admits well-defined large N and large D expansions for the enhanced scaling

More precisely, F is first expanded at large N

$$F = \sum_{h \in \mathbb{N}} F_h \, N^{2-h}$$

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Then, each F_h is expanded at large D

$$F_h = \sum_{\ell \in \mathbb{N}} F_{h,\ell} \, D^{r+h-\frac{\ell}{r+1}}$$

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Interesting features:

- large D limit governed by the **index**: $\ell = 2 \, \mathrm{ind}_0 \, \mathcal{B}$
- the two limits **don't** commute: $N \to \infty$ first, $D \to \infty$ second

Summary and outlook

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- New large N scaling for tensor models $\rightarrow$ well-defined and non-trivial large N expansion, governed by the index
- Larger class of leading order Feynman graphs than for usual scaling. In particular, able to capture the SYK-like physics

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Outlook

- Large N and D expansions for symmetric traceless matrix-tensor models (Klebanov, Tarnopolski, 2017; Benedetti, Carrozza, Gurau, Kolanowski - 2017)
- Characterization of the new family of leading order Feynman graphs

Discussion

- Gurau degree/ index $\rightarrow$ intrinsic, non-negative quantities associated to any colored graph
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- Index $\rightarrow$ 3-bubble subgraphs $\mathcal{B}_{(0ij)}$ $(i, j = 1, \ldots, R)$

$$\mathrm{ind}_0\, \mathcal{B} = \frac{1}{2} \sum_{i<j} h(\mathcal{B}_{(0ij)})$$

where $h(\mathcal{B}_{(0ij)})$ = quantity governing the large N expansion of the (ij) matrix model embedded in the tensor model

Discussion

- For $R = 2$, both reduce to the genus (usual $1/N$ expansion of matrix models)
- For $R \geq 3$, they are related by

$$\mathrm{ind}_0\, \mathcal{B} = \frac{1}{(R-2)!}\big(\deg \mathcal{B} - R \deg \mathcal{B}^{(0)}\big)$$

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- For $R \geq 3$, they are related by

$$\text{ind}_0 \mathcal{B} = \frac{1}{(R-2)!} \left(\deg \mathcal{B} - R \deg \mathcal{B}^{(0)} \right)$$

- Leading order graphs: satisfy $\text{ind}_0 \mathcal{B} = 0$ - **generalized melons** - and form a larger class than usual melons ($\deg \mathcal{B} = 0$)

Connection with high-energy physics

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1. SYK model (Sachdev, Ye - 1993; Kitaev - 2015)

   Quantum mechanical model of N Majorana fermions interacting through a random all-to-all interaction

   Interesting features:

   - Solvable at large N
   - Reparameterisation invariance in the IR regime
   - Maximally chaotic

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   But **not** a genuine quantum field theory

Connection with high-energy physics

2 SYK-like tensor models (Witten - 2016; Klebanov, Tarnopolsky - 2016)

Link between SYK model and some specific colored and uncolored tensor models

Interesting because:

- Same physics as SYK model at large N
- Well-defined quantum field theories

Note: same physics at leading order but different $1/N$ corrections (Bonzom, Lionni, Tanasa - 2017)

Connection with high-energy physics

3. String-inspired matrix models (Ferrari - 2017)

SYK model and SYK-like tensor models remain rather exotic from the point of view of the holographic correspondence

Best candidates: models with matrices of size $N \times N$ in the large N limit

Idea: introduction of a new parameter D to manage the large N expansion of matrix models

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$\longrightarrow$ **Matrix-tensor models**