1 Canonical realization of asymptotic symmetries

In this section, we answer the question of how to discriminate between trivial gauge transformations and asymptotic symmetries: the former lead to vanishing boundary charges, the latter to non-vanishing boundary charges.

Thus, the issue of finding the asymptotic symmetries for a given bulk theory with given boundary conditions is reduced to calculating the canonical boundary charges.

There are two main ways of calculating them, a covariant and a canonical way. We start with the former (in a slick manner) and then present the latter (to gain some additional intuition). A key feature is the possible emergence of central extensions in the canonical realization of the asymptotic symmetry algebra.

2 Covariant phase space summary

Here is a brief summary of the covariant phase space analysis of boundary charges in gauge theories; for more details and pertinent applications see e.g. chapter 1 in the lecture notes by Compère and Fiorucci or chapter 5 in our black holes book.

Take some Lagrangian, vary it, and use the equations of motion (EOM). You get a total derivative term,

$$\delta L[\phi] \approx \mathrm{d}\theta[\phi,\,\delta\phi] \tag{1}$$

where $\approx$ means going on-shell and $\delta\phi$ denotes some generic variation of a field ϕ on which the Lagrangian depends functionally. Taking the anti-symmetrized variation of the total derivative term θ yields the symplectic current

$$\omega[\phi,\,\delta_1\phi,\,\delta_2\phi] = \delta_1\theta[\phi,\,\delta_2\phi] - \delta_2\theta[\phi,\,\delta_1\phi] \tag{2}$$

whose integral over a co-dimension 1 hypersurface Σ yields the symplectic 2-form

$$\Omega[\phi,\,\delta_1\phi,\,\delta_2\phi] = \int_\Sigma \omega[\phi,\,\delta_1\phi,\,\delta_2\phi]\,. \tag{3}$$

There are two different form degrees at play here: there is the usual (spacetime) form degree, where L is a D form (a volume form in D spacetime dimensions), θ and ω are $D-1$ forms, and Ω is a 0-form. Additionally, we can attribute (anti-symmetrized) variations a form degree in the covariant phase space, i.e., $\delta\phi$, δL and θ are 1-forms whereas ω and Ω are 2-forms with respect to this form degree.

Infinitesimal gauge symmetries are local transformations of the form $\phi \to \phi + \delta_\varepsilon\phi$ with some local parameter $\varepsilon(x)$. The (variation of the) infinitesimal charge associated with such a transformation is defined as

$$\delta Q[\varepsilon] := \Omega[\phi,\,\delta\phi,\,\delta_\varepsilon\phi]\,. \tag{4}$$

Note that the right-hand side of (4) contains a generic field variation as well as a gauge transformation. A key result (proved by Wald and by Barnich and Brandt) is that the symplectic current for a gauge transformation is exact on-shell,

$$\omega[\phi,\,\delta\phi,\,\delta_\varepsilon\phi] \approx \mathrm{d}k_\varepsilon[\phi,\,\delta\phi]\,. \tag{5}$$

Inserting this result into the definition of the charges (4) by virtue of (3) together with Stokes' theorem yields

$$\delta Q[\varepsilon] = \oint_{\partial\Sigma} k_\varepsilon[\phi,\,\delta\phi]\,. \tag{6}$$

If it is possible to integrate the charges in field space (meaning that it is possible to get rid of the δ's on both sides of (6); in our examples, this always will be the case) then one obtains in this manner co-dimension two charges $Q[\varepsilon]$ associated with gauge symmetries generated by ε. The fact that the charges are co-dimension two is often called "Noether's second theorem", to discriminate from the Noether charges associated with global symmetries, which are co-dimension one.

Of course, it is possible that the charges defined by (6) vanish, depending on the specific physical system, the boundary conditions, and the gauge transformations that preserve them. If all the charges vanish then all transformations are gauge redundancies and one has no conserved boundary charges. If instead some of the charges are non-zero the associated transformations are no longer pure gauge at the boundary but rather generate asymptotic symmetries. We are finally in the position to define precisely what we mean by asymptotic symmetries:

Asymptotic symmetries are generated by all boundary condition-preserving transformations with gauge parameter ε, modulo gauge transformations whose associated boundary charges vanish, $Q[\varepsilon] = 0$.

Phrased differently, we can define proper gauge transformations as those that have $Q[\varepsilon] = 0$, while improper ones have $Q[\varepsilon] \neq 0$. Asymptotic symmetries are generated by all improper gauge transformations.

2.1 Example 0: free scalar

While there are no gauge-symmetries, consider as warm-up the free scalar Lagrangian D-form $L[\phi] = \frac{1}{2}\,\mathrm{d}\phi \wedge *\,\mathrm{d}\phi$. Its variation $\delta L[\phi,\,\delta\phi] = -\delta\phi\;\mathrm{d}{*}\,\mathrm{d}\phi + \mathrm{d}\big(\delta\phi * \mathrm{d}\phi\big)$ yields the Klein–Gordon EOM $*\,\mathrm{d}{*}\,\mathrm{d}\phi = 0$ (we dualized with $*$ to convert the volume form into a scalar) and the boundary form $\theta[\phi,\,\delta\phi] = \delta\phi * \mathrm{d}\phi$. This means the symplectic 2-form (w.r.t. covariant phase space; w.r.t. spacetime it is a 0-form)

$$\Omega(\delta_1\phi,\,\delta_2\phi) = \int_\Sigma \big(\delta_2\phi * \mathrm{d}\delta_1\phi - \delta_1\phi * \mathrm{d}\delta_2\phi\big) \tag{7}$$

is reminiscent of the mechanical symplectic 2-form $\delta_2 q\,\delta_1 p - \delta_1 q\,\delta_2 p$.

2.2 Example 1: electrodynamics

Take the Maxwell Lagrangian D-form $L[A] = -\frac{1}{4}\,F \wedge *F$ with $F = \mathrm{d}A$ featuring abelian gauge symmetries $\delta_\varepsilon A = \mathrm{d}\varepsilon$. Its variation $\delta L[A, \delta A] = \frac{1}{2}\,\delta A \wedge \mathrm{d}{*}\,\mathrm{d}F - \frac{1}{2}\,\mathrm{d}\big(\delta A \wedge *F\big)$ yields the Maxwell equations $\mathrm{d}{*}F = 0$ in the bulk and the boundary form $\theta[A,\,\delta A] = -\frac{1}{2}\delta A \wedge *F$. The symplectic 2-form

$$\Omega(\delta_1 A,\,\delta_2 A) = -\frac{1}{2}\int_\Sigma \big(\delta_2 A \wedge *\delta_1 F - \delta_1 A \wedge *\delta_2 F\big) \tag{8}$$

yields the (variation of) the boundary charge upon replacing δ_2 by a gauge variation and using the linearized EOM $\mathrm{d}{*}\delta F = 0$

$$\delta Q[\varepsilon] \approx \Omega(\delta A, \delta_\varepsilon A) = -\frac{1}{2}\int_\Sigma \mathrm{d}\varepsilon \wedge *\delta F = -\frac{1}{2}\int_{\partial\Sigma} \varepsilon * \delta F + \frac{1}{2}\int_\Sigma \varepsilon\;\mathrm{d}{*}\delta F \approx -\frac{1}{2}\int_{\partial\Sigma} \varepsilon * \delta F \tag{9}$$

establishing $k_\varepsilon(\delta A) = -\frac{1}{2}\,\varepsilon\; *\delta F$. For Coulomb gauge- and boundary conditions, $A = \frac{Q}{4\pi r}\;\mathrm{d}t$, $\delta A = \mathcal{O}(1/r)\;\mathrm{d}t$, the result above for $\varepsilon = 1$ (using $F_{tr} = -F_{rt} = \frac{Q}{4\pi r^2}$)

$$Q[\varepsilon = 1] = \frac{1}{2}\int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^{\pi} \sin\theta\,\mathrm{d}\theta\, r^2 (F_{tr} - F_{rt}) = Q \tag{10}$$

reproduces the expected Coulomb charge as the boundary charge (we fixed a sign).

2.3 Example 2: Chern–Simons

The Chern–Simons Lagrangian 3-form $L[A] = \frac{k}{4\pi}\,\langle A\wedge \mathrm{d}A + \frac{2}{3}\,A\wedge A\wedge A\rangle$ features non-abelian gauge symmetries $\mathrm{d}_\varepsilon A = \mathrm{d}\varepsilon + [A,\,\varepsilon]$. Its variation $\delta L[A,\,\delta A] = \frac{k}{2\pi}\,\langle F\wedge \delta A\rangle + \frac{k}{4\pi}\langle A\wedge\delta A\rangle$ yields the Chern–Simons EOM $F = \mathrm{d}A + A\wedge A = 0$ and the boundary form $\theta[A,\,\delta A] = \frac{k}{4\pi}\langle A\wedge\delta A\rangle$. The symplectic 2-form $\Omega(\delta_1 A,\,\delta_2 A) = \frac{k}{4\pi}\int_\Sigma\langle\delta_1 A\wedge\delta_2 A - \delta_2 A\wedge\delta_1 A\rangle$ yields $k_\varepsilon(\delta A) = \frac{k}{2\pi}\,\langle\varepsilon\,\delta A\rangle$ upon replacing δ_2 by a gauge variation and using the linearized EOM $\delta F = 0$.

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\int_{\partial\Sigma}\langle\varepsilon\,\delta A\rangle \tag{11}$$

If ε does not vary, $\delta\varepsilon = 0$, then the result (11) can be integrated in field space.

$$\boxed{Q[\varepsilon] = \frac{k}{2\pi}\int_{\partial\Sigma}\langle\varepsilon\,A\rangle} \tag{12}$$

The co-dimension-2 charges (12) feature prominently in our discussion of $\mathrm{AdS}_3/\mathrm{CFT}_2$.

3 Canonical summary and boundary charges

We are interested in systems with gauge symmetries in at least two spacetime dimensions, which in the canonical language means that we will have some first-class constraint(s) Φ generating these gauge transformations (to reduce clutter we do not decorate Φ with some counting index nor denote its dependence on phase space variables explicitly in this general part). Since we are dealing with field theories, it is useful to introduce smeared versions of constraints (integrating over some constant t-slice Σ),

$$G[\varepsilon] = \int_\Sigma \mathrm{d}^{D-1}x\,\varepsilon\,\Phi\,. \tag{13}$$

Functional variations of the smeared generators in general lead to boundary terms. When this happens, we say that the corresponding generator is not functionally differentiable.[1] In order to maintain functional differentiability we define improved generators that are equivalent to G in the bulk but differ by a boundary term in such a way that they are functionally differentiable,

$$\delta\Gamma[\varepsilon] = \delta G[\varepsilon] + \delta Q[\varepsilon] = \text{volume terms only}\,. \tag{14}$$

The quantity $\delta Q[\varepsilon]$ is the canonical version of the expression (6) and, when integrable in field space, is called "canonical boundary charge".

Here is a toy example. Suppose that we are in two spacetime dimensions and the only first-class constraint is given by $\Phi = f(\phi,\,\pi) + \pi\partial_x\phi + \alpha\phi\partial_x\pi$, with some function f and some constant α. Then the variation of the smeared constraint yields

$$\delta G[\varepsilon] = \int\limits_{x^0}^{x^1}\mathrm{d}x\Big(\varepsilon\big(\delta f + (\delta\pi)\,\partial_x\phi + \alpha(\delta\phi)\partial_x\pi\big) - \partial_x(\varepsilon\,\phi)\delta\phi - \alpha\partial_x(\varepsilon\,\phi)\,\delta\pi\Big)$$
$$+\,\varepsilon\,\big(\pi\delta\phi + \alpha\phi\,\delta\pi\big)\Big|_{x_0}^{x^1} = \text{volume terms} + \varepsilon\,\big(\pi\delta\phi + \alpha\phi\,\delta\pi\big)\Big|_{x_0}^{x^1}. \tag{15}$$

[1] Functional differentiability of the canonical gauge generators is the Hamiltonian analog of having a well-defined variational principle in the Lagrange formulation. For a review see 1312.6427.

The improved generator given by (14) requires the addition of a boundary term $\delta Q[\varepsilon]$ which can be read off from (15):

$$\delta Q[\varepsilon] = -\varepsilon \left(\pi\delta\phi + \alpha\phi\,\delta\pi\right)\Big|_{x_0}^{x^1} \tag{16}$$

Unless $\alpha = 1$ the expression (16) is not integrable in field space. For $\alpha = 1$ we succeed and obtain

$$Q[\varepsilon] = -\varepsilon\,\pi\phi\Big|_{x_0}^{x^1}\,. \tag{17}$$

The analysis leading to (14) is completely generic and background independent. In particular, it does not rely on the specification of particular boundary conditions on the fields. However, to address whether or not the expression $\delta Q[\varepsilon]$ can be integrated in field space to canonical boundary charges $Q[\varepsilon]$ does depend on the specific choice of boundary conditions. We assume henceforth that the charges are integrable. In practical applications, this has to be checked case by case, and we shall do so below for AdS_3 Einstein gravity with Brown–Henneaux boundary conditions. In order to perform such an analysis we need a couple of additional general results.

Readers unfamiliar with the Hamiltonian formulation should consult appendix A at this point before proceeding further.

3.1 Canonical gauge generators from Castellani algorithm

In the section above we were sketchy about the canonical gauge generators — we just said that they are smeared versions of first-class constraints, which is true, but did not specify *which* linear combination(s) of first-class constraints. We address this now.

Let us assume we have some field theory with a certain number of primary first-class constraints (PFC) and possibly additional second class and/or secondary first or second class constraints. We get rid of the second class constraints by introducing Dirac brackets and thus focus henceforth on a system of PFCs and possibly other (secondary) first-class constraints.

The canonical gauge generators are determined using Castellani's algorithm, which works as follows. The gauge generator is linear in the infinitesimal parameter ϵ and in its time derivatives,

$$G(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots, \partial_t^k \epsilon) = \sum_{n=0}^{k} \int_\Sigma \mathrm{d}^{D-1}x\,\Phi_n\,\partial_t^n \epsilon \tag{18}$$

where Φ_n is some linear combination of first-class constraints, such that the one with the largest index k is a PFC. The remaining contributions are determined recursively as follows.

$$\Phi_k = \text{PFC} \tag{19}$$

$$\Phi_{k-1} + \{\Phi_k,\,\mathcal{H}\} = \text{PFC} \tag{20}$$

$$\vdots$$

$$\Phi_0 + \{\Phi_1,\,\mathcal{H}\} = \text{PFC} \tag{21}$$

$$\{\Phi_0,\,\mathcal{H}\} = \text{PFC} \tag{22}$$

The physical reason why we need something like the Castellani algorithm and cannot just associate a gauge generator with each first-class constraint is that each Lagrangian gauge symmetry can (and in general does) correspond to more than one

Hamiltonian gauge symmetry, essentially because the Lagrangian formulation does not discriminate between taking time or spatial derivatives, while the Hamiltonian formulation does.

Let us check by means of a simple example, electrodynamics in D spacetime dimensions, why we need the Castellani algorithm (see appendix A.9 for more details). In electrodynamics we have the primary constraint $\pi^0 \approx 0$, the secondary ("Gauss") constraint $\psi = \partial_\alpha \pi^\alpha \approx 0$ and the total Hamiltonian density $\mathcal{H} = \frac{1}{2}\,\pi_\alpha\pi^\alpha + \frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} - A_0\psi$. What we want to achieve is to find a function G on phase space that produces the correct gauge transformations for A_μ via Poisson brackets:

$$\delta_\varepsilon A_0 = \{A_0,\, G[\varepsilon]\} = \dot\varepsilon \qquad\qquad \delta_\varepsilon A_\alpha = \{A_\alpha,\, G[\varepsilon]\} = \partial_\alpha\varepsilon \tag{23}$$

By inspection, you can see that the right choice is

$$G(\varepsilon,\dot\varepsilon) = \int_\Sigma \mathrm{d}^{D-1}x\,\big(\dot\varepsilon\,\pi^0 + (\partial_\alpha\varepsilon)\pi^\alpha + \dots\big) = \int_\Sigma \mathrm{d}^{D-1}x\,\big(\dot\varepsilon\,\pi^0 - \varepsilon\,\partial_\alpha\pi^\alpha + \dots\big) \tag{24}$$

where we neglect boundary terms. The Castellani algorithm produces precisely this result. The quantity Φ_1 above corresponds here to the only PFC we have, π^0. Then we have to solve the equations

$$\Phi_0 + \{\Phi_1,\, \mathcal{H}\} = a_0\Phi_1 \quad\Rightarrow\quad \Phi_0 + \partial_\alpha\pi^\alpha = a_0\pi^0 \tag{25}$$

where a_0 is undetermined at this stage, and

$$\{\Phi_0,\, \mathcal{H}\} = a_1\Phi_1 \quad\Rightarrow\quad a_0\psi = a_1\pi^0\,. \tag{26}$$

The last equality implies that both constants a_0 and a_1 vanish. Inserting these results into the general definition of the canonical gauge generator (18) yields

$$G(\epsilon,\dot\epsilon) = \int_\Sigma \mathrm{d}^{D-1}x\,\big(\dot\varepsilon\,\pi^0 - \varepsilon\,\partial_\alpha\pi^\alpha\big) \tag{27}$$

which coincides with the result (24) obtained by inspection, but now includes all boundary terms. Note that the canonical gauge generator for electrodynamics (27) is not functionally differentiable.

$$\delta G[\varepsilon] = \text{volume terms} - \oint_{\partial\Sigma} \mathrm{d}f_\alpha\,\varepsilon\,\delta\pi^\alpha \tag{28}$$

To repair this we define the improved generator (14) and find in this way the canonical boundary charges of electrodynamics

$$Q[\varepsilon] = \oint_{\partial\Sigma} \mathrm{d}f_\alpha\,\pi^\alpha\,\varepsilon \tag{29}$$

where we assumed that the parameter ε is state-independent. Since the spatial components of the canonical momentum correspond to the electric field we derived essentially the result (10). The canonical boundary charges (29) are co-dimension two quantities, in accordance with Noether's second theorem.

3.2 Canonical realization of asymptotic symmetry algebra

In the canonical language the asymptotic symmetry algebra is given by the quotient algebra of all boundary condition-preserving transformations modulo trivial gauge transformations that have vanishing canonical boundary charges.

As discussed in the previous section, the improved canonical generators of gauge symmetries are given by

$$\Gamma[\varepsilon] = G[\varepsilon] + Q[\varepsilon]\,. \tag{30}$$

If we denote going to the constraint surface by "weakly equal", ≈, then the relation (31) reduces to

$$\Gamma[\varepsilon] \approx Q[\varepsilon] \tag{31}$$

and shows that the improved canonical gauge generators reduce to the canonical boundary charges. This justifies calling $Q[\varepsilon]$ "boundary charges": they have support only at the boundary and give the value of the functionally differentiable gauge generator $\Gamma[\varepsilon]$ evaluated on the constraint surface. In other words, $Q[\varepsilon]$ is the (conserved) charge associated with the asymptotic symmetry associated with the transformation ε. Note that only the asymptotic behavior of ε plays a role and not its bulk behavior. This explains why we identify different transformations that preserve the boundary conditions but have the same asymptotic behavior for ε.

Gauge transformations of a function on phase space are generated by Poisson brackets of the improved canonical gauge generators with that function.

$$\delta_\varepsilon f = \{\Gamma[\varepsilon],\, f\} \tag{32}$$

If we take for the function f another gauge generator we obtain

$$\delta_{\varepsilon_1}\Gamma[\varepsilon_2] = \{\Gamma[\varepsilon_1],\, \Gamma[\varepsilon_2]\}\,. \tag{33}$$

Gauge-fixing and solving the constraints, the relation (33) can be re-expressed in terms of canonical boundary charges[2]

$$\delta_{\varepsilon_1} Q[\varepsilon_2] = \{Q[\varepsilon_1],\, Q[\varepsilon_2]\}\,. \tag{34}$$

Evaluating the brackets above on general grounds leads to an algebra

$$\{\Gamma[\varepsilon_1],\, \Gamma[\varepsilon_2]\} = \Gamma[\varepsilon_1 \circ \varepsilon_2] + Z[\varepsilon_1,\, \varepsilon_2] \tag{35}$$

that can have a central extension $Z[\varepsilon_1,\, \varepsilon_2]$. We discuss such extensions in the next subsection. The Dirac bracket algebra of the canonical boundary charges

$$\{Q[\varepsilon_1],\, Q[\varepsilon_2]\} = Q[\varepsilon_1 \circ \varepsilon_2] + Z[\varepsilon_1,\, \varepsilon_2] \tag{36}$$

has the same central extension.

The relations (34) and (36) are key for the canonical realization of the asymptotic symmetry algebra, which is the Poisson bracket algebra of the improved canonical generators, or equivalently the Dirac bracket algebra of the canonical boundary charges. Going into some basis $Q_n = Q[\varepsilon_n]$ (where ε_n are e.g. Fourier- or Laurent-modes of the function ε) this algebra generically takes the form

$$-\,i\{Q_n,\, Q_m\} = f_{nm}{}^k Q_k + Z_{nm} \tag{37}$$

where $f_{nm}{}^k$ are structure functions and Z_{nm} is a possible central extension of the algebra. Introducing the factor $-i$ is convenient for canonical quantization $-i\{,\} \to [,]$, where $[,]$ denotes commutators.

[2]The Poisson brackets should be replaced by Dirac brackets, which are nothing but the Poisson bracket on the reduced phase space. We will not discriminate in our notation Dirac from Poisson brackets as the meaning should always be clear from the context.

3.3 Central extensions and Virasoro example

Formally, a central extension z of a Lie algebra g (where z is in the center of the extended algebra e) is a short exact sequence[3]

$$0 \to z \to e \to g \to 0\,. \tag{38}$$

Central extensions of groups are defined analogously. An important result for verifying the existence of possible central extensions is that the set of isomorphism classes of central extensions of some group G by Z is in one-to-one correspondence with the second cohomology group $H^2(G,\,Z)$. In other words, central extensions are 2-cocycles. The definition above means that a centrally extended algebra can be written as a direct sum, $e = g \oplus z$.

In our applications, we are interested in commutator, Poisson- or Dirac-bracket algebras and their possible central extensions. In this language, a central extension of such an algebra is the addition of terms that commute with all generators and that do not spoil the Jacobi identities. We demand central extensions to be non-trivial, i.e., they cannot be eliminated by a change of basis.

Here is a pertinent example. Take for g the Witt algebra

$$[L_n,\,L_m] = (n-m)\,L_{n+m} \tag{39}$$

and consider a possible central extension to an algebra e

$$[L_n,\,L_m] = (n-m)\,L_{n+m} + \mathbb{1}\,Z(n,m)\,. \tag{40}$$

The new generator $\mathbb{1}$ is part of the center z and by definition commutes with all Witt generators L_n. To simplify the notation it is often dropped and we shall do the same. The function $Z(n,\,m)$ has to be anti-symmetric in n and m to be consistent with antisymmetry of the bracket. Let us now check the Jacobi identities

$$\begin{aligned}[][[L_n,\,L_m],\,L_k] + \text{cycl}(n,m,k) &= [(n-m)\,L_{n+m} + Z(n,m),\,L_k] + \text{cycl}(n,m,k)\\ &= (n-m)\,(n+m-k)\,L_{n+m+k} + (n-m)\,Z(n+m,k) + \text{cycl}(n,m,k)\\ &= (n-m)\,Z(n+m,k) + \text{cycl}(n,m,k) \overset{!}{=} 0 \qquad \forall n,m,k\in\mathbb{Z}\,. \end{aligned}\tag{41}$$

While there are (infinitely) many functions $Z(n,m)$ that solve (41), most of them are related by a redefinition of the generators $L_n \to L_n + \mathbb{1}\,A(n)$. Under such a redefinition this function transforms as $Z(n,m) \to Z(n,m) - (n-m)\,A(n+m)$. Let us now choose specifically (and with no loss of generality) $A(n) = Z(n,0)/n$ for non-vanishing n and $A(0) = Z(1,-1)/2$. The transformed function $Z(n,m)$ obeys the relations

$$Z(n,0) = Z(0,m) = Z(1,-1) = 0 \qquad \forall n,m\,. \tag{42}$$

The particular brackets

$$[L_n,\,L_0] = n\,L_0 \qquad\qquad [L_1,\,L_{-1}] = 2L_0 \tag{43}$$

have no central extension in this basis (which guarantees that the sl$(2,\mathbb{R})$ subalgebra generated by $L_{\pm1}$, L_0 has no central extension). Consider now the Jacobi identities (41) for $k=0$ and $n+m\neq 0$.

$$[[L_n,\,L_m],\,L_0] + \text{cycl}(n,m,0) = (m+n)\,Z(m,n) = 0 \qquad n\neq -m \tag{44}$$

[3]If you are unfamiliar with exact sequences do not worry too much. The arrows denote homomorphisms, and the exactness refers to the fact that the image of each homomorphism is the kernel of the next homomorphism. So the image of the homomorphism from the central extension z to the extended algebra e is the kernel of the homomorphism of the extended algebra e to the Lie algebra g.

This means that the function $Z(n, m)$ can only be non-zero if $n = -m$,

$$Z(n, m) = z(n)\, \delta_{n+m,\,0} \qquad z(n) = -z(-n) \qquad z(1) = 0\,. \tag{45}$$

To determine the function $z(n)$ we consider the Jacobi identities (41) for $k = 1$

$$[[L_n,\, L_m],\, L_1] + \text{cycl}(n, m, 1) = (n+2)\, z(n) + (1-n)\, z(n+1) = 0 \tag{46}$$

which establish the recursion relation

$$z(n+1) = \frac{n+2}{n-1}\, z(n) \quad \Rightarrow \quad z(n) = \frac{(n+1)n(n-1)(n-2)(n-3)\dots 4}{(n-2)(n-3)\dots 4\cdot 3\cdot 2\cdot 1}\, z(2) \tag{47}$$

that is solved by

$$z(n) = \frac{z(2)}{6}\, n\, (n^2-1)\,. \tag{48}$$

It is conventional to denote $z(2) = \frac{c}{2}$. Thus, we finally have established the unique (up to changes of basis) central extension of the Witt algebra known as **Virasoro algebra with central charge c**.

$$\boxed{[L_n,\, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, (n^3 - n)\, \delta_{n+m,\,0}} \tag{49}$$

4 Chern–Simons boundary charges

Let us now implement the general results above for Chern–Simons theories on cylinders, $\mathcal{M} = \mathbb{R}\times\Sigma$, where Σ is a disk. Since we use the canonical formulation we start with a 2+1 split in the action

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^2x\, \epsilon^{\alpha\beta}\, \mathfrak{g}_{ab}\, \big(\dot{A}^a_\alpha A^b_\beta + A^a_0 F^b_{\alpha\beta} + \partial_\alpha(\dots)\big) \tag{50}$$

where $\mathfrak{g}_{ab} = \text{tr}(T_a T_b)$ is determined by the bilinear form on our Lie algebra, a, b denote Lie-algebra indices with respect to some basis T_a, dot denotes partial derivative with respect to time t, the two-dimensional Levi–Civitá symbol is defined by $\epsilon^{\alpha\beta} = \epsilon^{t\alpha\beta}$, and α, β are spatial indices. The field strength components read explicitly

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + f^a{}_{bc}\, A^b_\alpha A^c_\beta \tag{51}$$

where $f^a{}_{bc}$ are the structure constants of the Lie algebra, $[T_a,\, T_b] = f^c{}_{ab}\, T_c$.

We follow now the usual canonical recipe, see for instance the textbook by Henneaux and Teitelboim and appendix A. Our full phase space is spanned by A^a_μ and their momenta $\pi^\mu_a = \partial\mathcal{L}/\partial\dot{A}^a_\mu$ (Greek indices from the middle of the alphabet range over all spacetime indices), which have canonical Poisson brackets

$$\{A^a_\mu(x),\, \pi^\nu_b(y)\} = \delta^a_b\, \delta^\nu_\mu\, \delta^{(2)}(x-y)\,. \tag{52}$$

The Legendre transformation from Lagrange to Hamilton is singular and we have the following (primary) constraints

$$\Phi^0_a = \pi^0_a \approx 0 \qquad \Phi^\alpha_a = \pi^\alpha_a - \frac{k}{4\pi}\, \epsilon^{\alpha\beta}\mathfrak{g}_{ab} A^b_\beta \approx 0\,. \tag{53}$$

The canonical Hamilton density (dropping total derivative terms) is given by

$$\mathcal{H}_{\text{can}} = -\frac{k}{4\pi}\, \epsilon^{\alpha\beta}\mathfrak{g}_{ab}\, A^a_0 F^b_{\alpha\beta} \tag{54}$$

The total Hamiltonian density, which generates our time evolution, is given by the sum of the canonical one and the primary constraints,

$$\mathcal{H} = \mathcal{H}_{\text{can}} + \lambda^a_\mu \, \Phi^\mu_a \tag{55}$$

where λ^a_μ are Lagrange multipliers.

Consistency of the primary constraints demands that their time evolution vanishes weakly, $\dot{\Phi}^\mu_a = \{\Phi^\mu_a, \, \mathcal{H}\} \approx 0$, which leads to the conditions

$$\bar{\psi}_a = -\frac{k}{4\pi} \, \epsilon^{\alpha\beta} \mathfrak{g}_{ab} \, F^b_{\alpha\beta} \approx 0 \qquad\qquad \mathcal{D}_\alpha A^a_0 - \lambda^a_\alpha \approx 0 \tag{56}$$

with the gauge covariant derivative $\mathcal{D}_\alpha X^a := \partial_\alpha X^a + f^a{}_{bc} A^b_\alpha X^c$. Thus, we have a new (secondary) constraint $\bar{\psi}_a$. No further constraints arise since $\dot{\bar{\psi}}_a \approx 0$. Shifting the constraint $\bar{\psi}_a \to \psi_a = \bar{\psi}_a - \mathcal{D}_\alpha \Phi^\alpha_a$ the Hamiltonian density can be written as sum over constraints

$$\mathcal{H} = A^a_0 \, \psi_a + \lambda^a_0 \, \Phi^0_a \approx 0 \,. \tag{57}$$

The fact that the Hamiltonian vanishes weakly is a typical feature of theories that are reparametrization invariant.

We need to determine which constraints are first-class and second class, so we look at their Poisson brackets. The non-vanishing ones are given by

$$\{\Phi^\alpha_a(x), \, \Phi^\beta_b(y)\} = -\frac{k}{2\pi} \, \epsilon^{\alpha\beta} \mathfrak{g}_{ab} \, \delta^{(2)}(x-y) \tag{58}$$

$$\{\Phi^\alpha_a(x), \, \psi_b(y)\} = -f_{ab}{}^c \, \Phi^\alpha_c \, \delta^{(2)}(x-y) \approx 0 \tag{59}$$

$$\{\psi_a(x), \, \psi_b(y)\} = -f_{ab}{}^c \, \psi_c \, \delta^{(2)}(x-y) \approx 0 \,. \tag{60}$$

Therefore, Φ^0_a and ψ_a are first-class constraints, while Φ^α_a are second class constraints.

We can now count the number of local physical degrees of freedom: per point and per Lie algebra generator we have a six-dimensional unphysical phase space (spanned by A^a_μ and π^μ_a), two first-class constraints (Φ^0_a and ψ_a) and two second class constraints (Φ^α_a) so that the dimension of the physical phase space is zero. This confirms the expectations we had from our Lagrangian discussion of Chern–Simons theories where we found that all solutions to the field equations are locally pure gauge. To eliminate the second class constraints we replace the Poisson brackets with Dirac brackets, which turn out to be identical to the former, with the only exception $\{A^a_\alpha(x), \, A^b_\beta(y)\} = \frac{2\pi}{k} \, \mathfrak{g}^{ab} \epsilon_{\alpha\beta} \, \delta^{(2)}(x-y)$.

Applying Castellani's algorithm yields again a canonical gauge generator that contains a $\dot{\varepsilon}$-term multiplied by our primary first-class constraint and an ε-term multiplied by our secondary first-class constraint (everything is decorated by Lie-algebra index a, but otherwise the result is analogous to electrodynamics).

$$G[\varepsilon^a] = \int_\Sigma \mathrm{d}^2x \, \big((D_0 \varepsilon^a) \pi^a_0 + \varepsilon^a \psi_a\big) \tag{61}$$

As in electrodynamics, the generator (61) is not functionally differentiable,

$$\delta G[\varepsilon^a] = \text{volume terms} - \oint_{\partial\Sigma} \mathrm{d}f_\alpha \, \big(\frac{k}{4\pi} \, \epsilon^{\alpha\beta} \mathfrak{g}_{ab} \, \epsilon^a \, \delta A^b_\beta + \epsilon^a \, \delta\pi^\alpha_a\big) \tag{62}$$

which we remedy again by introducing an improved generator

$$\delta\Gamma[\varepsilon^a] = G[\varepsilon^a] + \delta Q[\varepsilon^a] \tag{63}$$

that differs from the previous one by a total derivative term

$$\delta Q[\varepsilon^a] = \frac{k}{2\pi}\int_\Sigma \mathrm{d}^2x\,\partial_\alpha\big(\epsilon^{\alpha\beta}\mathfrak{g}_{ab}\,\epsilon^a\,\delta A^b_\beta\big) \tag{64}$$

Using Stokes' theorem we rewrite the result above succinctly as

$$\boxed{\delta Q[\varepsilon] = \frac{k}{2\pi}\oint_{S^1}\langle\varepsilon\,\delta A\rangle} \tag{65}$$

This is our main result, which establishes the canonical boundary charges for arbitrary Chern–Simons theories on a cylinder, regardless of the specific boundary conditions.

It is worthwhile stressing that whether or not the charges defined by (65) are integrable in field space, conserved in time, finite and non-trivial (all desirable properties for a physical theory) depends very much on the precise boundary conditions, which is why a case-by-case analysis is needed from this point on. The parameters ε appearing in the variation of the canonical boundary charges have to preserve the boundary conditions that specify the behavior of allowed fluctuations δA, viz.

$$\delta_\varepsilon A = \mathrm{d}\varepsilon + [A,\,\varepsilon] \stackrel{!}{=} \mathcal{O}(\delta A)\,. \tag{66}$$

In the next section, we consider a specific example of relevance for $\mathrm{AdS}_3/\mathrm{CFT}_2$.

Before doing so let us quickly consider what happens when we take the adapted ansatz $A = b^{-1}(\mathrm{d}{+}a)b$ for the connection, where $\delta b = 0$. If we similarly split $\varepsilon = b^{-1}\hat\varepsilon b$ and use $\delta A = b^{-1}\delta a b$ as well as cyclicity of the bilinear form, $\langle b^{-1}ABb\rangle = \langle ABbb^{-1}\rangle = \langle AB\rangle$, then the result (65) reduces to

$$\delta Q[\hat\varepsilon] = \frac{k}{2\pi}\oint_{S^1}\langle\hat\varepsilon\,\delta a\rangle\,. \tag{67}$$

If a is chosen such that it is independent of the "radial" coordinate the result (67) is independent of this radial coordinate and thus manifestly finite as r tends to its asymptotic value. This is one of the reasons why this specific ansatz for the connection is so useful: it makes the canonical charges (when they exist) manifestly finite and independent from any radial cut-off surface. Also the conditions (66) for a transformation to be boundary condition-preserving simplify

$$\delta_{\hat\varepsilon} a = \mathrm{d}\hat\varepsilon + [a,\,\hat\varepsilon] \stackrel{!}{=} \mathcal{O}(\delta a)\,. \tag{68}$$

5 Three-dimensional Einstein gravity

We are finally in a position to make precise statements about the physical phase space of three-dimensional Einstein gravity with various boundary conditions. For concreteness — and also since it is probably the most interesting example for holography — we focus on Brown–Henneaux boundary conditions in AdS_3 Einstein gravity using the Chern–Simons formulation in terms of two $\mathrm{sl}(2,\,\mathbb{R})$ connections $A^\pm$.

$$A^\pm = b_\pm^{-1}\left(\mathrm{d}{+}a^\pm\right)b_\pm \qquad\qquad b_\pm = e^{\pm\rho/\ell\,L_0} \tag{69}$$

$$a^\pm = \big(L_{\pm1} - \mathcal{L}^\pm(x^\pm)\,L_{\mp1}\big)\,\frac{\mathrm{d}x^\pm}{\ell} \qquad\Rightarrow\qquad \delta a^\pm = -\delta\mathcal{L}^\pm(x^\pm)\,L_{\mp1}\,\frac{\mathrm{d}x^\pm}{\ell}\,. \tag{70}$$

Our first task is to determine the boundary condition-preserving transformations generated by some $\varepsilon^\pm = b_\pm^{-1}\hat\varepsilon^\pm b_\pm$. To simplify the notation we focus on the upper sign equations and drop all $\pm$ and $\mp$ decorations since all formulas are analogous for both signs. We also set the AdS radius to unity, $\ell = 1$, decompose $\hat{\varepsilon}$ algebraically

$$\hat{\varepsilon} = \varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1} \tag{71}$$

and solve (68) in order to find all boundary condition-preserving transformations.

$$\big(\varepsilon(x)'\,L_1 + \varepsilon_0(x)'\,L_0 + \varepsilon_{-1}(x)'\,L_{-1}\big)\,\mathrm{d}x + [L_1,\,\varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1}]\,\mathrm{d}x \\ - [\mathcal{L}(x)\,L_{-1},\,\varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1}]\,\mathrm{d}x \stackrel{!}{=} -\delta\mathcal{L}(x)\,L_{-1}\,\mathrm{d}x \tag{72}$$

The L_1 component of this equation yields

$$\varepsilon_0(x) = -\varepsilon(x)'\,. \tag{73}$$

The L_0 component obtains

$$\varepsilon_{-1}(x) = \frac{1}{2}\,\varepsilon(x)'' - \mathcal{L}(x)\varepsilon(x)\,. \tag{74}$$

The L_{-1} component does not lead to any restriction of the functions ε_n since $\delta\mathcal{L}$ is an arbitrary function. It is still useful to see the explicit expression, after inserting the results (73) and (74):

$$\boxed{\delta_\varepsilon\mathcal{L} = 2\mathcal{L}\,\varepsilon' + \mathcal{L}'\,\varepsilon - \frac{1}{2}\,\varepsilon'''} \tag{75}$$

The function $\mathcal{L}$ transforms with an infinitesimal Schwarzian derivative.

We have just derived the variation of the canonical boundary charges (67) for Brown–Henneaux boundary conditions,

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\oint_{S^1} \varepsilon\,\delta\mathcal{L}\,\mathrm{d}x\,. \tag{76}$$

Assuming that ε is state-independent and hence has vanishing variation, $\delta\varepsilon = 0$, the charges (67) are integrable in field space and we can drop the δ's on both sides.

Reinstating the $\pm$-decorations we thus find two non-trivial towers of canonical boundary charges for AdS_3 Einstein gravity with Brown–Henneaux boundary conditions:

$$\boxed{Q^\pm[\varepsilon^\pm(x^\pm)] = \frac{k}{2\pi}\oint_{S^1} \varepsilon^\pm(x^\pm)\,\mathcal{L}^\pm(x^\pm)\,\mathrm{d}x^\pm} \tag{77}$$

Our next task is to establish the canonical realization of the asymptotic symmetries, i.e., to derive the asymptotic symmetry algebra as Poisson bracket algebra generated by the charges (77). This is important since the physical phase space falls into representations of that algebra, so it is valuable information to discover what that algebra is. To this end, we exploit (34), which relates the brackets of two charges to the variation of one charge with respect to the parameter of the other charge. This means that for the determination of the asymptotic symmetry algebra, we only need to know the general variation of the charges for any boundary conditions-preserving transformation — which we just have derived! Inserting (75) into (76) together with (34) establishes

$$\{Q[\varepsilon_1],\,Q[\varepsilon_2]\} = Q[\varepsilon_1'\varepsilon_2 - \varepsilon_2'\varepsilon_1] - \frac{k}{4\pi}\oint_{S^1} \varepsilon_1'''\varepsilon_2\,\mathrm{d}x\,. \tag{78}$$

Intriguingly, the asymptotic symmetry algebra (78) comes with a central extension (the last term; see also (36)).

In order to see more explicitly which algebra this is we introduce Fourier modes for all functions. This is possible due to our assumption of the theory living on a cylinder. We define the Fourier mode generators

$$L_n^\pm := Q[e^{inx^\pm}] \tag{79}$$

and multiply (78) by $-i$ to obtain our final result for the asymptotic symmetry algebra.

$$\boxed{-i\{L_n^\pm,\, L_m^\pm\} = (n-m)\, L_{n+m}^\pm + \frac{k}{2}\, n^3\, \delta_{n+m,\,0}} \tag{80}$$

Comparing (80) with (49) we see that upon replacing $-i\{,\} \to [,]$ (which is canonical quantization) the asymptotic symmetry algebra consists of two copies of the Virasoro algebra with central charge

$$\boxed{c = 6k = \frac{3\ell}{2G}\,.} \tag{81}$$

If you are wondering what happened to the term linear in n in the central extension in (49): we can recover it from (80) by shifting the zero mode generator $L_0 \to L_0 + k/4 = L_0 + c/24$ (remember that the precise form of a central extension is basis dependent). This shift of the Virasoro zero mode by $c/24$ is well-known in the CFT_2 literature and can be interpreted as the Casimir energy of the cylinder.

This derivation (albeit in the metric formulation) was first performed by Brown and Henneaux in 1986 and proves that **AdS_3 Einstein gravity for Brown–Henneaux boundary conditions is equivalent to a CFT_2**, in the sense that the physical phase space (or upon quantization the physical Hilbert space) falls into representations of two copies of the Virasoro algebra, which is the conformal algebra in two dimensions. In retrospect, this result was a milestone on the road to the AdS/CFT correspondence (this paper currently has more than 2200 citations according to INSPIRE; however, in the first decade after publication it was widely unknown — a classic example of a "sleeper" paper).

A Hamiltonian analysis

This appendix is intended for readers who are unfamiliar with the Hamiltonian formulation of classical mechanics or field theory and can be skipped otherwise. I assume familiarity with the Lagrangian formulation.

A.1 Phase space

In geometric terms, the Lagrangian picture relies on the tangent bundle of configuration space. The latter is described by a set of (generalized) coordinates q, which provide a snapshot of a dynamical system at any given time (we suppress any indices on q — the configuration space can have any dimension, including infinity). The Lagrangian $L(q, \dot{q})$ depends not only on the generalized coordinates but also its velocities $\dot{q}$, hence the tangent bundle. It is then natural to seek a formulation in terms of the dual to the tangent bundle, i.e., the cotangent bundle. For convex Lagrangians, this is achieved by a Legendre transformation,

$$H_{\text{can}}(q,\, p) := p\dot{q} - L(q,\, \dot{q}) \qquad\qquad p := \frac{\partial L(q,\, \dot{q})}{\partial \dot{q}} \tag{82}$$

since Legendre transformations transform functions on a vector space to functions on the dual vector space and are involutive. The phase space is spanned by the generalized coordinates q and their associated momenta p. The subscript "can" means "canonical". The function H is called "canonical Hamiltonian" and is an important function on the phase space.

Geometrizing the Hamiltonian formulation leads to symplectic geometry, but we shall be as light on the notation and concepts as possible, so actually all we need for later purposes are canonical Poisson brackets. They are defined for any differentiable functions on phase space $F(q,p)$, $G(q,p)$:

$$\{F(q,p),\, G(q,p)\} := \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} \tag{83}$$

By construction, the Poisson brackets are anti-symmetric and obey the Jacobi identities $\{\{A,\, B\},\, C\} + \{\{B,\, C\},\, A\} + \{\{C,\, A\},\, B\} = 0$. Unless something goes wrong with the Legendre transformation (82), the Hamiltonian defined therein generates the time evolution of any function on phase space via the Poisson brackets (83)

$$\dot{F}(q,p) = \{F(q,p),\, H(q,p)\}\,. \tag{84}$$

This could be the end of the appendix if it were not for certain singularities in the Legendre transformation that arise in gauge systems.

A.2 Singular Legendre trafo and primary constraints

If all accelerations $\ddot{q}$ can be expressed uniquely in terms of coordinates q and velocities $\dot{q}$ the Legendre transformation from Lagrangian to Hamiltonian is fine, so consider now instead what happens when this is not the case. This means that the matrix $\partial^2 L/(\partial q \partial q)$ does not have full rank, which implies that we get a number of constraints that equals the dimension of the kernel of this matrix. They are called "primary constraints" and can be expressed as functions on the phase space that vanish,

$$\Phi_i(q,p) \approx 0 \qquad\qquad i = 1..\text{dim ker}\; \partial^2 L/(\partial q \partial q)\,. \tag{85}$$

The notation $\approx$ means "weakly equal" and always refers to equality up to a linear combination of constraints. The picture of the phase space is now modified: not the full phase space is physical, but only a hypersurface within the phase space defined by the vanishing of all constraints. This hypersurface is called "constraint surface".

A.3 Total Hamiltonian

The time evolution is no longer described by the canonical Hamiltonian (82) since it fails to take into account the primary constraints (85). The correct time evolution is instead generated by the total Hamiltonian

$$H(q,p;\lambda) = H_{\text{can}}(q,p) + \lambda^i \Phi_i(q,p) \tag{86}$$

where λ is a set of Lagrange multipliers (there are as many as there are primary constraints). Variation of the total Hamiltonian with respect to the Lagrange multipliers λ puts us on the constraint surface defined by (85). By construction, the Hamiltonian equations of motion following from (86) with (84) are equivalent to the original Euler–Lagrange equations of motion following from the original Lagrangian $L(q,\dot{q})$.

A.4 Secondary constraints

Time evolution should be consistent with restricting to the constraint surface, but at this stage, we cannot expect such a consistency. So we need to consider the time evolution of the primary constraints (85).

$$\dot{\Phi}_i = \{\Phi_i,\, H\} = \{\Phi_i,\, H_{\text{can}}\} + \lambda^j \{\Phi_i,\, \Phi_j\} \tag{87}$$

There are five possibilities at this stage. For each i the condition (86) can

1. be inconsistent
2. be fulfilled as strong equality for any choice of λ^j
3. be fulfilled as weak equality for any choice of λ^j
4. be fulfilled as strong or weak equality for a specific choice of λ^j
5. lead to new constraints $\psi_i(q,p) \approx 0$

In the first case, the theory is mathematically inconsistent and must be discarded. In cases 2.-4. the algorithm stops. In case 5. the new constraints are called "secondary constraints" and the algorithm reboots, in the sense that we have to check for consistency of the time evolution of these new constraints analogous to (87). In this way successively more and more secondary constraints can arise (sometimes called "ternary", "quaternary" etc., and sometimes all of them are referred to as "secondary"). If we started with a finite dimensional phase space this procedure of generating new constraints necessarily stops at some point. When this happens we have a complete set of constraints that is compatible with the time evolution generated by the total Hamiltonian (86).

A.5 First class constraints and gauge transformations

This is not yet the end of the story. There are two different classes of constraints: first-class constraints have (weakly) vanishing Poisson brackets with all other constraints, otherwise they are second class. Because of this property, the consistency relations (87) do not determine the Lagrange multipliers of first-class constraints. In other words, while for second class constraints the Lagrange multipliers are fixed to certain values, for first-class constraints they are arbitrary. This leads to the following: **First class constraints generate gauge transformations.** By "gauge transformations" we mean a redundancy in the physical description. It is easy to see why this is true: since the λ are arbitrary the Hamiltonians $H_1 = H_{\text{can}} + \lambda_1\Phi$ and $H_2 = H_{\text{can}} + \lambda_2\Phi$ must generate the same time evolution. Their difference, $H_1 - H_2 = (\lambda_1 - \lambda_2)\,\Phi$ must then generate what we called gauge transformation. Gauge transformations of arbitrary functions F on phase space generated by a first-class constraint Φ are defined by

$$\delta_\varepsilon F(q,p) := \varepsilon\,\{F(q,p),\, \Phi(q,p)\}\,. \tag{88}$$

A.6 Second class constraint and Dirac bracket

While our main focus is on first-class constraints let us briefly discuss how to treat second class constraints χ_i. By definition, they must have non-vanishing Poisson brackets so that the matrix

$$\{\chi_i,\, \chi_j\} = C_{ij} \tag{89}$$

has full rank, which means there is always an even number of second class constraints. Let us denote the inverse matrix of C_{ij} by C^{ij}. Then the Dirac bracket is defined by

$$\{F,\, G\}_{\text{D}} := \{F,\, G\} - \{F,\, \chi_i\}\, C^{ij}\, \{\chi_j,\, G\} \tag{90}$$

and has the following properties: like the Poisson bracket it is antisymmetric and obeys the Jacobi identity; the first-class property is preserved when going from

Poisson to Dirac brackets; most importantly, the Dirac bracket with any second class constraint vanishes identically. Thus, the Dirac bracket puts us on the part of the constraint surface defined by all the second class constraints. It is equivalent to the Poisson bracket of a reduced phase space, where all weak equalities involving second class constraints are converted into strong equalities, $\approx \to =$.

Here is a prototypical example. Take the Lagrangian $L = q_1\dot{q}_2 - V(q_1, q_2)$. The canonical momenta are given by $p_2 = q_1$ and $p_1 = 0$. The matrix $\partial L/(\partial\dot{q}_i\partial\dot{q}_j)$ vanishes, so the dimension of its kernel is two. Hence we get two primary constraints:

$$\chi_1 = p_2 - q_1 \approx 0 \qquad\qquad \chi_2 = p_1 \approx 0 \tag{91}$$

The total Hamiltonian is given by

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - q_1\dot{q}_2 + V + \tilde{\lambda}^1\,\chi_1 + \tilde{\lambda}^2\,\chi_2 = V + \lambda^1\,\chi_1 + \lambda^2\,\chi_2\,. \tag{92}$$

The consistency relations

$$\dot{\chi}_1 = \{\chi_1,\,H\} = \{p_2 - q_1,\,H\} = -\frac{\partial V}{\partial q_2} - \lambda^2 \tag{93}$$

$$\dot{\chi}_2 = \{\chi_2,\,H\} = \{p_1,\,H\} = -\frac{\partial V}{\partial q_1} + \lambda^1 \tag{94}$$

allow to solve for $\lambda^{1,2}$ and hence lead to no secondary constraints. Thus, we have only two (primary) constraints $\chi_{1,2}$. Since their Poisson bracket does not vanish weakly

$$\{\chi_i,\,\chi_j\} = C_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad\Rightarrow\quad C^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{95}$$

they are both second class. The Dirac bracket (90) is then given by

$$\{F,\,G\}_{\text{D}} := \{F,\,G\} - \{F,\,\chi_1\}\,\{\chi_2,\,G\} + \{F,\,\chi_2\}\,\{\chi_1,\,G\} \tag{96}$$

leading to

$$\{q_1,\,F\}_{\text{D}} = \frac{\partial F}{\partial p_1} + \{p_2 - q_1,\,F\} = -\frac{\partial F}{\partial q_2} \qquad\qquad \{p_1,\,F\}_{\text{D}} = 0 \tag{97}$$

$$\{q_2,\,F\}_{\text{D}} = \frac{\partial F}{\partial p_2} - \{p_1,\,F\} = \frac{\partial F}{\partial p_2} + \frac{\partial F}{\partial q_1} \qquad\qquad \{p_2,\,F\}_{\text{D}} = -\frac{\partial F}{\partial q_2} \tag{98}$$

showing that the reduced phase space is two-dimensional (p_2 is redundant with q_1 and p_1 is trivial). For functions $F(q_1, q_2)$ that only depend on the canonical coordinates (we can assume this with no loss of generality since the constraints eliminate both momenta) the equations above are precisely canonical Poisson brackets for a single degree of freedom if we identify q_1 with p and q_2 with q.

We could have made our lives simpler from the very beginning just by relabeling $q_1 \to p$ and $q_2 \to q$, in which case the Lagrangian directly would lead to the Hamiltonian of the reduced phase space, $L = p\dot{q} - V(q, p)$. The Dirac algorithm above is a somewhat byzantine way of arriving at the same result. While more lengthy, it does have the feature that you need not think, but can blindly apply it.

A.7 Physical phase space

The dimension of the physical phase space is in general smaller than the dimension of the full phase space due to the presence of constraints. Each constraint reduces its dimension by one, and each first-class constraint additionally reduces its dimension by one due to gauge redundancies. Thus, the dimension of the physical phase space equals the dimension of the full phase space minus the number of second class constraints minus twice the number of first-class constraints.

$$\text{dim physical} = \text{dim full} - \#(2^{\text{nd}}\text{ class}) - 2 \times \#(1^{\text{st}}\text{ class}) \tag{99}$$

The dimension of the physical phase space is even, since there is always an even number of 2^{nd} class constraints and the dimension of the full phase space is also even. Half of the dimension of the physical phase space is what we usually call the number of physical degrees of freedom. In field theories, all these statements are understood pointwise.

A.8 Canonical gauge generator

Every first-class constraint generates a gauge transformation acting on phase space functions via (88). However, not all of them correspond to different gauge symmetries from a Lagrangian viewpoint. In order to determine the number of Lagrangian gauge symmetries it is sufficient to count the number of first-class primary constraints (PFCs).[4] Secondary first-class constraints descending from a given PFC then generate the same gauge symmetry from a Lagrangian perspective, but with additional time derivatives of the transformation parameter, since these constraints arose from considering time derivatives of PFCs.

For each PFC we define the canonical gauge generator as

$$G(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots, \partial_t^k \epsilon) = \sum_{n=0}^{k} \int_\Sigma \mathrm{d}^{D-1}x \, \Phi_n \, \partial_t^n \epsilon \tag{100}$$

where the number k depends on the stage where the consistency algorithm stops. For instance, if the PFC does not lead to any secondary constraint then $k = 0$. If it leads to a single secondary constraint then $k = 1$. If it leads to a secondary constraint that in turn generates another secondary (a.k.a. ternary) constraint then $k = 2$ and so on. The constraint with the highest index, Φ_k, is then nothing but our PFC we started with. The linear combination of constraints with next-to-highest index, Φ_{k-1}, is determined by the time-evolution of this PFC, up to a linear combination of PFCs. This pattern continues, i.e., Φ_{n-1} is determined by the time-evolution of Φ_n, until we reach Φ_0. The final relation is then that the time evolution of Φ_0 should be a linear combination of PFCs. This procedure leads to the Castellani algorithm summarized in section 3.1.

In many applications the Dirac algorithm stops after one stage, i.e., for $k = 1$, including electrodynamics, Yang–Mills theories, the Standard Model, Einstein gravity, Chern–Simons theories, and BF-theories. We shall discuss electrodynamics as the prototypical example of such gauge theories in section A.9 below.

Having said all this it is perfectly fine to ignore the Castellani algorithm and associate a canonical gauge generator with each first-class constraint, regardless of whether it is primary or secondary. The key aspect of the canonical gauge generator regarding functional differentiability and boundary contributions does not depend on which of these routes is taken.

[4]Note that the distinction between primary and secondary constraints is not a profound one but depends on the formulation. Nonetheless, in most applications, this distinction is useful, as we shall see in the case of electrodynamics discussed at the end of this appendix. The view taken in the textbook by Henneaux and Teitelboim is to treat all first-class constraints on equal footing, which is fine if your focus is exclusively a Hamiltonian one.

A.9 Electrodynamics as example

The Maxwell action ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$S[A_\mu] = \int \mathrm{d}^D x \, \mathcal{L} = -\frac{1}{4} \int \mathrm{d}^D x \, F_{\mu\nu} F^{\mu\nu} \tag{101}$$

is the prototypical example of a gauge theory. The explicit form of the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu} = -\frac{1}{2} \left(\partial^0 A^\mu - \partial^\mu A^0\right) \tag{102}$$

shows that we have a primary constraint

$$\Phi = \pi^0 \approx 0 \,. \tag{103}$$

Up to boundary terms, the canonical Hamiltonian obtained by Legendre transforming (101) reads

$$\mathcal{H}_{\text{can}} = \frac{1}{2} \, \pi^\alpha \pi_\alpha + \frac{1}{4} \, F_{\alpha\beta} F^{\alpha\beta} - A_0 \partial_\alpha \pi^\alpha \tag{104}$$

where α, β range over $1..(D-1)$.

The primary constraint (103) is preserved in time

$$\dot{\Phi} = \{\Phi, \, \mathcal{H}_{\text{can}}\} = \{\pi^0, \, -A_0 \partial_\alpha \pi^\alpha\} = \partial_\alpha \pi^\alpha \overset{!}{=} 0 \tag{105}$$

only of the secondary constraint (known as "Gauss constraint")

$$\psi = \partial_\alpha \pi^\alpha \approx 0 \tag{106}$$

holds. The Gauss constraint is preserved in time, $\dot{\psi} = \{\psi, \, \mathcal{H}_{\text{can}}\} \approx 0$, and thus does not generate any further constraints.

Having found all the constraints, we check whether they are first or second class. Since all relevant Poisson brackets vanish,

$$\{\Phi, \, \psi\} = 0 \tag{107}$$

both constraints are first-class. Therefore, we have $2D - 4$ canonical degrees of freedom, which translates into $D - 2$ Lagrangean degrees of freedom, a.k.a. photon polarizations.

To continue this example in the presence of boundaries, go back to section 3.1.

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