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OIST Exercises on Asymptotic symmetries

(2.1) **Boundary charges for the Hilbert action of Einstein gravity**
Determine the boundary charges associated with diffeomorphisms for Einstein gravity, starting with the Hilbert action in $D > 2$ dimensions.

(2.2) **Constraint analysis of Chern–Simons theory**
Start with the 3-dimensional Chern–Simons bulk action

$$I_{\text{CS}}[A] = \frac{k}{4\pi} \int \mathrm{d}^3x \, \epsilon^{\mu\nu\lambda} \, h_{ab} \big(A^a_\mu \partial_\nu A^b_\lambda + \tfrac{1}{3} \, f^a{}_{cd} \, A^c_\mu A^d_\nu A^b_\lambda \big)$$

where k is the Chern–Simons level, $A = T_a A^a_\mu \, \mathrm{d}x^\mu$ a Lie-algebra valued connection 1-form [for concreteness you may use $sl(2, \mathbb{R})$ for the Lie-algebra], $f^a{}_{bc}$ are the corresponding structure constants, i.e., $[T_a, T_b] = f^c{}_{ab} T_c$ and h_{ab} is the Cartan–Killing metric, $h_{ab} = \langle T_a, T_b \rangle$, where T_a are the Lie-algebra generators. Your main task is to perform a Hamiltonian analysis of the theory, determining all primary, secondary and higher constraints, as well as their class (first/second). Use your results to count the number of local physical degrees of freedom.

(2.3) **Violation of Jacobi identities**
Consider Chern–Simons theory on a cylinder, $\mathcal{M} = \mathbb{R} \times \mathcal{D}$, with Dirichlet boundary conditions $A^a_0|_{\partial\mathcal{M}} = 0$ at the timelike boundary, $r = r_0$, where r is some radial coordinate. Take the (functionally non-differentiable) gauge generators (i refers to the two spatial directions)

$$G[\xi] = \frac{k}{4\pi} \int_{\mathcal{D}} \mathrm{d}^2x \, \xi^a \, \epsilon^{ij} h_{ab} F^b_{ij} \approx 0$$

and show that the Jacobi identities

$$\{G[\xi_1], \{G[\xi_2], G[\xi_3]\}\} + \text{cycl}(1,2,3) \stackrel{?}{=} 0$$

do not hold on-shell (due to a boundary term) if the ξ^a_n are chosen as follows: $\xi^a_1 = \rho \, A^a_r$ (with $\rho = 1$ near $\partial\mathcal{M}$ and $\rho = 0$ near the origin), $\xi^a_2 = \eta^a$ (with $\eta^a|_{\partial\mathcal{M}} = 0$), and $\xi^a_3 = \xi^a$ is unconstrained. [For further explanations on the notations see the previous exercise.]

Hints:

- I leave it to you whether you want to do this using the covariant phase space method or canonical methods. In the former case, you can compare your results with section 1.4.5 in `1801.07064`.

- Proceed as you would in any other gauge theory: define your canonical coordinates, derive the canonical momenta, determine the primary constraints, use the Dirac consistency algorithm to determine secondary constraints, check whether there are ternary (or higher) constraints and then determine the first/second class properties of all constraints. Once you have all these data assembled use the standard counting of the dimension of the physical phase space in the presence of first and second class constraints. If you want to compare your results with the literature see for instance appendix A in https://arxiv.org/abs/1209.2860.

- If you have solved the previous exercise you can use these results to quickly solve this exercise. If not, you will need the result that the Dirac bracket between two spatial connection components is given by

$$\big\{A_i^a(x),\, A_j^b(y)\big\} = \frac{2\pi}{k}\, h^{ab}\, \epsilon_{ij}\, \delta^{(2)}(x-y)\,.$$

  See `1312.6427` for a review and section 2 therein for the solution to this exercise.