Preservation of concavity properties by the Dirichlet heat flow and applications

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It is well known that heat flow preserves the log-concavity of the initial datum, in the following sense: if $\phi \geq 0$ is log-concave (i.e., $\log \phi$ is concave), and u is the (bounded) solution of $u_t = \Delta u$ in $R^n \times (0, +\infty)$ with $u(x, 0) = \phi$, then $u(\cdot, t)$ is log-concave for every $t \geq 0$.

Together with Ishige and Takatsu, we investigated on the optimality of this property and considered the more general concept of F-.concavity, discovering that, in a suitable sense, log-concavity is the weakest concavity property preserved by the heat flow, while the strongest is what we call "hot concavity". For our investigation we use only pdes techniques, while the original proof of the preservation of log-concavity by the heat flow, due to Brascamp and Lieb, is easily obtained as an application of a functional-geometric inequality known as Prekòpa-Leindler inequality. It is interesting to notice that is is also possible to do the way back, retrieving PL inequality (and the whole family opf Borell-Brascamp-Lieb inequalities) thanks to the concavity preservation properties of parabolic equations, so establishing a perfect equivalence between these two apparently separated worlds. This investigation was done in collaboration with Ishige and Liu.