On a sharp interface limit of the Kobayashi-Warren-Carter system

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The Kobayashi-Warren-Carter (KWC) system is a phase field model which was introduced around 2000 to describe evolution of structures of each grain of multi-grain systems in materials science. In a simple model, unknowns are an order parameter v and a structure variable u like (averaged) angle of crystal. A typical example of energy is of the form

$$E^{\varepsilon}_{\mathrm{KWC}}(u,v) := \int v^2 |Du| + E^{\varepsilon}_{\mathrm{sMM}}(v),$$

$$E^{\varepsilon}_{\mathrm{sMM}}(v) := \varepsilon \int \frac{|\nabla v|^2}{2}\, dx + \frac{1}{2\varepsilon} \int (v-1)^2\, dx,$$

where the integral is taken over a domain in $\mathbb{R}^n$. Here $\varepsilon > 0$ is a small parameter corresponding to interface thickness and $|Du|$ denote the total variation measure which agrees with $|\nabla u|$ of the gradient ∇u of u if Du is absolutely continuous with respect to the Lebesgue measure.

The KWC system is roughly speaking an L^2-gradient flow of $E^{\varepsilon}_{\mathrm{KWC}}$. Its global well-posedness itself is nontrivial because the energy contains total variation. Such problem has been studied by a group of N. Kenmochi and others and currently actively studied by K. Shirakawa (Chiba University).

In this talk, we discuss a few topics related to $E^{\varepsilon}_{\mathrm{KWC}}$. The first topic is the singular limit of $E^{\varepsilon}_{\mathrm{KWC}}$ as ε tends to zero which is often called a sharp interface limit. We are able to characterize its limit by characterizing the singular limit of a single-well Modica-Mortola type energy $E^{\varepsilon}_{\mathrm{sMM}}$. The second topic is the singular limit of the gradient flow with a suitable scaling. It turns out that we have a gradient flow involving fractional time derivative on a grain boundary at least in one-dimensional setting when u is independent of time. As the third topic, we also discuss minimizers of a fidelity term flow total variation type energy which is obtained by minimizing the limit of $E^{\varepsilon}_{\mathrm{KWC}}$ with order parameter. It turns out that all minimizer must be piecewise constant in one-dimensional setting.

We also discuss a similar energy like the Ambrosio-Tortorelli approximation of the Mumford-Shah energy. This talks is based on several joint works with A. Kubo (Hokkaido U.), H. Kuroda (Hokkaido U.), J. Okamoto (Kyoto U.), K. Sakakibara (Kanazawa U./RIKEN) and M. Uesaka (DetaLabs Inc.).