# Microlocal analysis of $d$-plane transform on the Euclidean space

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# Contents

# 1-1. $d$-plane transform

- $n = 2, 3, 4, \dots, d = 1, \dots, n-1$.
- The Grassmannian $G_{d,n}$ is the set of all $d$-dimensional vector subspaces of $\mathbb{R}^n$. $\dim G_{d,n} = d(n-d)$.
- The affine Grassmannian $G(d,n)$ is the set of all $d$-dimensional planes in $\mathbb{R}^n$, that is, $G(d,n) = \{x'' + \sigma \;:\; \sigma \in G_{d,n}, x'' \in \sigma^\perp\}$.
  $N(d,n) := \dim G(d,n) = (d+1)(n-d)$.
  $x'' + \sigma$ is sometimes denoted by $(\sigma, x'')$.
- Denote $x = x' + x'' \in \sigma \oplus \sigma^\perp = \mathbb{R}^n$. The $d$-plane transform of $f(x) = f(x' + x'') = \mathcal{O}(\langle x \rangle^{-d-\varepsilon})$ is defined by

$$\mathcal{R}_d f(\sigma, x'') = \int_\sigma f(x' + x'') dx', \tag{1}$$

  where $\langle x \rangle = \sqrt{1 + |x|^2}$ and $dx'$ is the Lebesgue measure on $\sigma$.
- $\mathcal{R}_1 f$ is said to be the X-ray transform of $f$,
  and $\mathcal{R}_{n-1} f$ is said to be the Radon transform $f$.

# 1-2. Filtered back-projection

The formal adjoint of $\mathcal{R}_d$ is given by

$$\begin{aligned}\mathcal{R}_d^*\varphi(x) &= \frac{1}{C(d,n)}\int_{\{\Xi\in G(d,n):x\in\Xi\}}\varphi(\Xi)\,d\mu(\Xi)\\ &= \frac{1}{C(d,n)}\int_{O(n)}\varphi(x+k\cdot\sigma)\,dk,\end{aligned}$$

where $x\in\mathbb{R}$, $\varphi\in C\bigl(G(d,n)\bigr)$,
$C(d,n)=(4\pi)^{d/2}\Gamma(n/2)/\Gamma\bigl((n-d)/2\bigr)$, $d\mu$ and $dk$ are normalized measure, and $\sigma\in G_{d,n}$.

**Proposition 1 (FBP (filtered back-projection))**

*For $f(x)=\mathcal{O}(\langle x\rangle^{-d-\varepsilon})$,*

$$f=(-\Delta_x)^{d/2}\mathcal{R}_d^*\mathcal{R}_d f=\mathcal{R}_d^*(-\Delta_{x''})^{d/2}\mathcal{R}_d f, \tag{2}$$

*where $-\Delta_x=-\partial_{x_1}^2-\cdots-\partial_{x_n}^2$, and $-\Delta_{x''}$ is the Laplacian on $\sigma^\perp$.*

# 1-3. Range of the $d$-plane transform

## Proposition 2

*$\mathcal{R}_d : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}_H\big(G(d,n)\big)$ is bijective, where $\mathscr{D}_H\big(G(d,n)\big)$ is the set of all $\varphi \in \mathscr{D}\big(G(d,n)\big)$ with the following conditions: for any $k = 0, 1, 2, \dots$, there exists a homogeneous polynomial $P_k$ on $\mathbb{R}^n$ of degree $k$ such that for any $\sigma \in G_{d,n}$,*

$$P_k(\xi'') = \int_{\sigma^\perp} \varphi(\sigma, x'')(\xi'' \cdot x'')^k dx'', \quad \xi'' \in \sigma^\perp.$$

Note that if $f(x) \in \mathscr{D}(\mathbb{R}^n)$, then we can define $P_k(\xi)$ by

$$P_k(\xi) := \int_{\mathbb{R}^n} f(x)(\xi \cdot x)^k dx, \quad \xi \in \mathbb{R}^n.$$

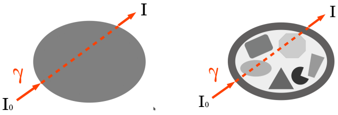
For any $\sigma \in G_{d,n}$ and $\xi'' \in \sigma^\perp$,

$$P_k(\xi'') = \int_{\sigma^\perp} \left( \int_\sigma f(x' + x'') dx' \right) (\xi'' \cdot x'')^k dx''.$$

# 1-4. CT scanner

Consider the following situation on $\mathbb{R}^2$:

- Here is an object whose attenuation coefficient distribution is $\boldsymbol{f}(\boldsymbol{x})$.
- The X-ray beam is supposed to have no width, and traverses the object along a line $\gamma$. $\boldsymbol{I_0}$ and $\boldsymbol{I}$ denote the intensities of the beam before and after passing through the object respectively.

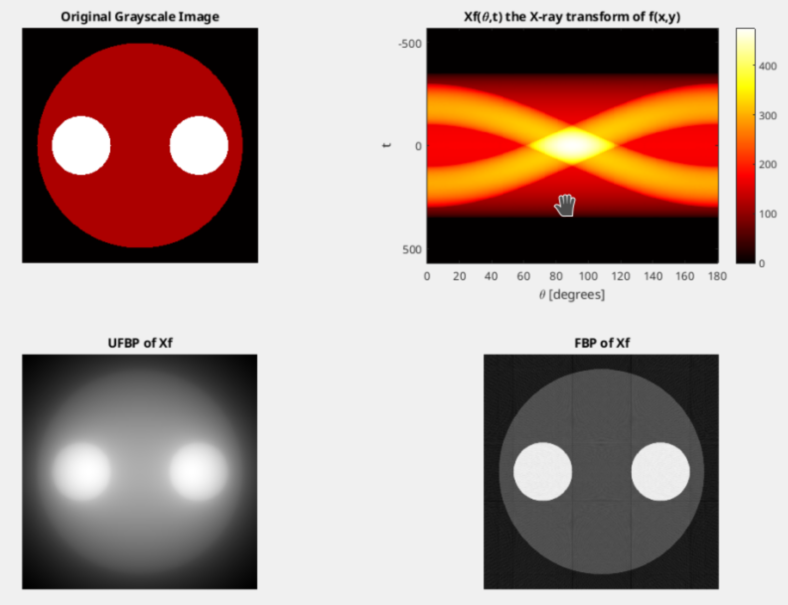



If the object is uniform, that is, $\boldsymbol{f} = \boldsymbol{d} \cdot \chi_\Omega$ and the travel length in the object is $\ell$, then the Beer-Lambert law obtains

$$\log\left(\frac{I_0}{I}\right) = d \cdot \ell = \int_\gamma f = \mathcal{R}_1 f(\gamma).$$

The same formula can be obtained for more general $\boldsymbol{f}$, and we can regard $\mathcal{R}_1 \boldsymbol{f}(\gamma)$ as the measurement of CT scanners.

# 1-5. MATLAB illustrates $\mathcal{R}_1$, UFBP and FBP.



# 1-6. Beam hardening

- There are some factors causing artifacts in CT images: beam width, partial volume effect, beam hardening, noise in measurements, numerical errors and etc.
- In the formulation of CT scanners in Page 6, the X-ray is supposed to be monochromatic with a fixed energy, say $E_0 > 0$.
- Actually, however, the X-ray beam has a wide range of energy $E$ and the attenuation coefficient distribution $f_E$ depends on $E$. This is described by the spectral function $\rho(E)$ which is a probability density function of $E \in [0, \infty)$. The formulation of the measurements of CT scanners becomes

$$\log\left(\frac{I_0}{I}\right) = -\log\left\{\int_0^\infty \rho(E)\exp(-\mathcal{R}_1 f_E)\,dE\right\}.$$

If $f_E$ is independent of $E$, that is, $f_E = f_{E_0}$, then

$$\log\left(\frac{I_0}{I}\right) = -\log\left\{\exp(-\mathcal{R}_1 f_{E_0}) \cdot \int_0^\infty \rho(E)\,dE\right\} = \mathcal{R}_1 f_{E_0}.$$

# 1-7. Metal streaking artifacts

- Consider a simple model of the beam hardening:

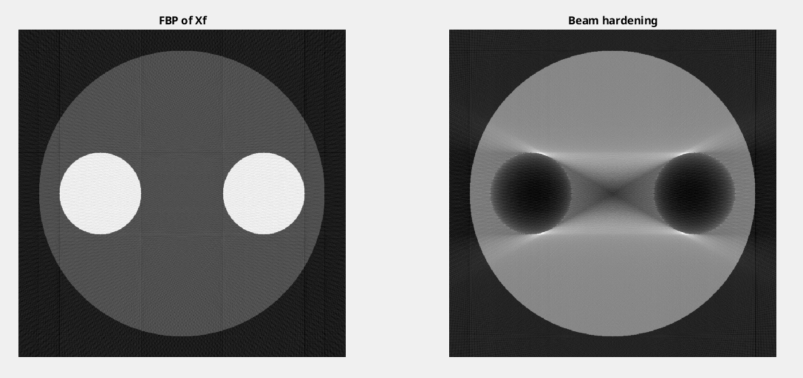
$$\rho(E) = \frac{1}{2\epsilon}\chi_{[E_0-\epsilon, E_0+\epsilon]}(E),$$
$$f_E(x) = f_{E_0}(x) + \alpha(E - E_0)\chi_D(x),$$

where $f_{E_0}$ is an attenuation coefficient distribution of human tissue, $\varepsilon$ and $\alpha$ are small positive constants, and $D$ is a metal region. Then the measurement becomes

$$P = \mathcal{R}_1 f_{E_0} - \log\left\{\frac{\sinh(\alpha\varepsilon\mathcal{R}_1\chi_D)}{\alpha\varepsilon\mathcal{R}_1\chi_D}\right\}$$
$$= \mathcal{R}_1 f_{E_0} + \sum_{k=1}^{\infty} A_k(\alpha\varepsilon\mathcal{R}_1\chi_D)^{2k}.$$

- Park-Choi-Seo (CPAM, 2017) proved that the metal streaking artifacts occur and they are described in the wave front set.
- Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) proved that the streaking artifacts are conormal distributions.

# 1-8. MATLAB illustrates metal streaking artifacts.




- Left: $\mathcal{R}_1^*(-\Delta_{x''})(\mathcal{R}_1 f_{E_0} + \alpha\varepsilon\mathcal{R}_1\chi_D)$.
- Right: $\mathcal{R}_1^*(-\Delta_{x''})\left\{\mathcal{R}_1 f_{E_0} - \frac{1}{3}(\alpha\varepsilon\mathcal{R}_1\chi_D)^2\right\}$.

# 2-1. Wave front set of distributions

**Definition 3 (wave front set)**

Let $X$ be a manifold. For $u \in \mathscr{D}'(X)$ and $(x,\xi) \in T^*X \setminus 0$, we say that $(x,\xi) \notin \mathrm{WF}(u)$ if there exists $\phi \in C_0^\infty(X)$ with $\phi(y) \neq 0$ near $y = x$ and a conic neighborhood of $V$ at $\eta = \xi$ such that

$$\widehat{\phi u}(\eta) = \mathcal{O}(\langle \eta \rangle^{-M}) \quad \text{for} \quad \eta \in V, \quad M = 1,2,3,\dots .$$

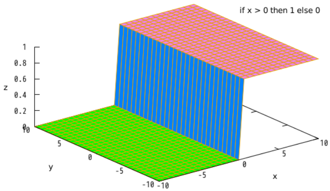
In other words,

$$\Sigma_x(u) := \bigcap_{\phi(x)\neq 0} \{\eta \in T_x^* X \setminus 0 : \widehat{\phi u}(\eta) \neq \mathcal{O}(\langle \eta \rangle^{-\infty})\},$$

$$\mathrm{WF}(u) := \{(x,\xi) \in T^*X \setminus 0 : \xi \in \Sigma_x(u)\}.$$

# 2-2. The characteristic function of a half-plane

Set $f(x,y) = 1$ if $x > 0$, $0$ otherwise in $\mathbb{R}^2$:




Let $\phi(x,y) \in \mathscr{D}(\mathbb{R}^2)$.

- If $\phi(0,y) = 0$, then $\widehat{\phi f}(\xi,\eta) = \hat{\phi}(\xi,\eta)$ or $\widehat{\phi f}(\xi,\eta) = 0$. We have $\widehat{\phi f}(\xi,\eta) = \mathcal{O}(\langle \xi;\eta\rangle^{-\infty})$, where $\langle \xi;\eta\rangle = \sqrt{1+|\xi|^2+|\eta|^2}$.
- If $\phi(0,y) \neq 0$, then

$$\widehat{\phi f}(\xi,\eta) = \text{const} \cdot \mathbf{pv} \int_{\mathbb{R}} \frac{\hat{\phi}(\zeta,\eta)}{\xi - \zeta} d\zeta.$$

- Then $\mathbf{WF}(f) = \{(0,y,\xi,0) \mid y,\xi \in \mathbb{R}, \xi \neq 0\}$.

# 2-3. Conormal distributions

## Definition 4 (Conormal distributions)

Let $X$ be an $N$-dimensional manifold, and let $Y$ be a closed submanifold of $X$. $u \in \mathscr{D}'(X)$ is said to be conormal with respect to $Y$ of degree $m$ if

$$L_1 \cdots L_M u \in {}^\infty H^{\mathrm{loc}}_{(-m-N/4)}(X)$$

for all $M = 0, 1, 2, \dots$ and all vector fields $L_1, \dots, L_M$ tangential to $Y$. Denote by $I^m(X, N^*Y)$ or $I^m(N^*Y)$, the set of all distributions on $X$ conormal with respect to $Y$ of degree $m$.

$$\|u\|_{{}^\infty H_{(s)}(\mathbb{R}^N)} := \sup_{j=0,1,2,\dots} \left( \int_{X_j} \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2},$$

$$X_0 := \{|\xi| < 1\}, \qquad X_j := \{2^{j-1} \leqq |\xi| < 2^j\}, \quad (j = 1, 2, 3, \dots).$$

# 2-4. Conormal distributions and oscillatory integrals

## Proposition 5 (Characterization of conormal distributions)

*Let $X$ be an $N$-dimensional manifold, and let $Y$ be a closed submanifold of $X$ with $\operatorname{codim} Y = k$. Fix arbitrary $y \in Y$ and choose local coordinates $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that $x(y) = 0$ and $Y$ is represented $\{x' = 0\}$ locally. For $u \in I^{m+k/2-N/4}(N^*Y)$ there exists an amplitude $a(x'', \xi') \in S^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ such that*

$$u(x) = \int_{\mathbb{R}^k} e^{ix'\cdot\xi'} a(x'', \xi') d\xi' \quad \text{near} \quad x = 0.$$

*Here $S^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ is the set of all smooth functions on $\mathbb{R}^{N-k} \times \mathbb{R}^k$ such that for any compact set $K$ in $\mathbb{R}^{N-k}$ and multi-indices $\alpha, \beta$*

$$|\partial_{x''}^{\beta} \partial_{\xi'}^{\alpha} a(x'', \xi')| \leqq C_{K\alpha\beta} \langle \xi \rangle^{m-|\alpha|} \quad \text{for} \quad (x'', \xi') \in K \times \mathbb{R}^k$$

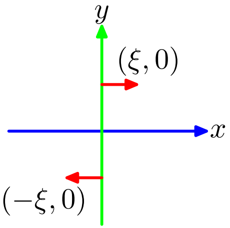
*with some positive constant $C_{K\alpha\beta}$.*

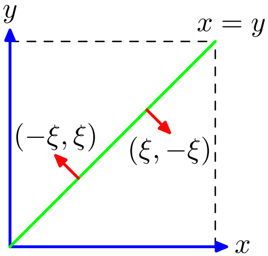
# 2-5. Examples of conormal distributions

- $f(x,y) \in I^{-1}(N^*(\{x=0\}))$.
- If $a(x,\xi) \in S^m(\mathbb{R}^N \times \mathbb{R}^N)$, then

$$K(x,y) = \int_{\mathbb{R}^N} e^{i(x-y)\cdot\xi} a(x,\xi)\,d\xi \in I^m(N^*\Delta),$$

where $\Delta = \{(x,x) : x \in \mathbb{R}^N\}$.



# 2-6. non-degenerate phase functions

## Definition 6 (non-degenerate phase functions)

Let $X$ be an $N$-dimensional manifold. We say that $\varphi(x,\theta) \in C^\infty(X\times\mathbb{R}^k \setminus \{0\};\mathbb{R})$ is a non-degenerate phase function if

- $\varphi(x,t\theta) = t\varphi(x,\theta)$ for $(x,\theta) \in X\times\mathbb{R}^k \setminus \{0\}$ and $t > 0$.
- $d\varphi(x,\theta) \neq 0$.
- If $\varphi'_\theta(x,\theta) = 0$, then $\operatorname{rank}[d\varphi'_\theta(x,\theta)] \equiv k$.

In this case $C_\varphi = \{(x,\theta) : \varphi'_\theta(x,\theta) = 0\}$ is an $N$-dimensional submanifold of $X \times \mathbb{R}^k \setminus \{0\}$. Set $\Lambda_\varphi := \left\{\left(x,\varphi'_x(x,\theta)\right) : \varphi'_\theta(x,\theta) = 0\right\}$. Then

$$C_\varphi \ni (x,\theta) \mapsto \left(x,\varphi'_x(x,\theta)\right) \in \Lambda_\varphi$$

is a diffeomorphism and $\Lambda_\varphi$ becomes a conic Lagrangian submanifold of $T^*X \setminus 0$, that is, a conic submanifold with $d\xi\wedge dx \equiv 0$.

# 2-7. Lagrangian distributions

The notion of conormal distributions is generalized as follows.

**Definition 7 (Lagrangian distribution)**

Let $\Lambda$ be a conic Lagrangian submanifold of $T^*X \setminus 0$.
$I^m(X,\Lambda) = I^m(\Lambda)$ is the set of all $u \in \mathscr{D}'(X)$ satisfying

- $\mathrm{WF}(u) \subset \Lambda$.
- For any $(x_0, \xi_0) \in \Lambda$ there exist a non-degenerate phase function $\varphi(x,\theta)$ and an amplitude $a(x,\theta) \in S^{m+N/4-k/2}(X \times \mathbb{R}^k)$ such that $\Lambda = \Lambda_\varphi$ and

$$u(x) = \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta \quad \text{near} \quad (x_0, \xi_0).$$

# 2-8. Fourier integral operators

Let $X$ and $Y$ be manifolds, and let $\Lambda$ be a conic Lagrangian submanifold of $T^*(X\times Y)\setminus 0$. For $K(x,y)\in I^m(X\times Y,\Lambda)$,

$$Au(x) := \int_Y K(x,y)u(y)dy, \quad u\in \mathscr{D}(Y)$$

defines a linear operator of $\mathscr{D}(Y)$ to $\mathscr{E}(X)$. $A$ is said to be a Fourier integral operator. Locally $A$ is given by

$$Au(x) := \iint e^{i\varphi(x,y,\theta)}a(x,y,\theta)u(y)dyd\theta.$$

with some $\varphi(x,y,\theta)$ and $a(x,y,\theta)$. If the principal part of $a$ does not vanish, then $A$ is called an elliptic FIO.

$\Lambda' := \{(x,y;\xi,\eta) : (x,y;\xi,-\eta)\in\Lambda\}$ is said to be the canonical relation of $A$. We have $\mathbf{WF}(Au)\subset\Lambda'\circ\mathbf{WF}(u)$ and if $A$ is elliptic then $\mathbf{WF}(Au)=\Lambda'\circ\mathbf{WF}(u)$ holds.

# 3-1. $T^*G(d,n)$

## Lemma 8

*$T^*G(d,n)$ is expressed as*
$\{(\sigma, x''; \eta''_1, \dots, \eta''_d, \xi'') : \sigma \in G_{d,n}, x'', \eta''_1, \dots, \eta''_d, \xi'' \in \sigma^\perp\}$.

**Proof.** Fix an arbitrary $(\sigma, x'') \in G(d,n)$. The set of (co-)tangent vectors corresponding to $\sigma$ is a $d(n-d)$-dimensional vector space since $\sigma \in G_{d,n}$. There exists an orthonormal system $\{\omega_1, \dots, \omega_d\} \subset \mathbb{S}^{n-1}$ such that $\sigma = \mathbf{span}\langle\omega_1, \dots .\omega_d\rangle$. If we replace $\omega_j$ by $c\omega_j$ with $c \in \mathbb{R} \setminus \{0\}$, $\sigma$ does not change. Hence (co-)tangent vector $\eta_j$ corresponding to $\omega_j$ belongs to $\omega_j^\perp$. Since $\sigma$ is independent of the choice of $\{\omega_1, \dots, \omega_d\} \subset \mathbb{S}^{n-1}$, (co-)tangent vectors $\eta_1, \dots, \eta_d$ corresponding to $\sigma$ belong $\omega_1^\perp \cap \dots \cap \omega_d^\perp = \sigma^\perp$. □

**Notation.** The orthogonal projection of $\mathbb{R}^n$ onto $\sigma \in G_{d,n}$ is denoted by $\pi_\sigma$.

# 3-2. The canonical relation of the *d*-plane transform

## Theorem 9

*$\mathcal{R}_d$ is an elliptic Fourier integral operator whose distribution kernel belongs to*

$$I^{0+(n-d)/2-(N(d,n)+n)/4}\big(G(d,n)\times\mathbb{R}^n;\Lambda_\phi\big),$$

$$m+\frac{k}{2}-\frac{N}{4}=0+\frac{n-d}{2}-\frac{N(d,n)+n}{4}=-\frac{d(n-d+1)}{4}.$$

$$\begin{aligned}\Lambda_\phi' &= \{(\sigma, y-\pi_\sigma y, y; \pm|\pi_\sigma u|\eta, 0,\dots,0,\eta,\eta) : \\ &\qquad \sigma\in G_{d,n}, y\in\mathbb{R}^n, \eta\in\sigma^\perp\} \\ &= \{(\sigma, x'', x''+t\omega; t\xi, 0,\dots,0,\xi,\xi) : \\ &\qquad (\sigma,x'')\in G(d,n), t\in\mathbb{R}, \omega\in\sigma\cap\mathbb{S}^{-1}, \xi\eta\in\sigma^\perp\}.\end{aligned}$$

# 3-3. Fourier slice theorem

For any $\sigma \in G_{d,n}$ and $\xi \in \sigma^{\perp}$,

$$\int_{\sigma^\perp} e^{-ix''\cdot\xi}\mathcal{R}_d f(\sigma, x'')dx'' = \int_{\sigma^\perp}\int_{\sigma} e^{-ix''\cdot\xi} f(x'+x'')dx'dx''$$
$$= \int_{\sigma^\perp}\int_{\sigma} e^{-i(x'+x'')\cdot\xi} f(x'+x'')dx'dx'' = \hat{f}(\xi'').$$

The for any $(\sigma, x'') \in G(d,n)$,

$$\begin{aligned}\mathcal{R}_d f(\sigma, x'') &= \frac{1}{(2\pi)^{n-d}}\int_{\sigma^\perp} e^{ix''\cdot\xi}\hat{f}(\xi'')d\xi \\ &= \frac{1}{(2\pi)^{n-d}}\int_{\sigma^\perp}\int_{\mathbb{R}^n} e^{i(x''-y)\cdot\xi} f(y)dyd\xi \\ &= \frac{1}{(2\pi)^{n-d}}\int_{\sigma^\perp}\int_{\mathbb{R}^n} e^{i\phi(\sigma,x'',y,\xi)} f(y)dyd\xi,\end{aligned}$$

$$\phi(\sigma, x'', y, \xi) = (x''-y)\cdot\xi, \quad (\sigma, x'') \in G(d,n), y \in \mathbb{R}^n, \xi \in \sigma^\perp.$$

It suffices to show that $\phi$ is a non-degenerate phase function.

## 3-4. $\phi(\sigma, x'', y, \eta)$ is a phase function.

- Critical points $\phi'_\xi = 0$:
  For $((\sigma, x''), y, \Xi) \in G(d,n) \times \mathbb{R}^n \times \mathbb{R}^n$, set $\Psi_1(\sigma, x'', y, \Xi) := \phi(\sigma, x'', y, \Xi - \pi_\sigma \Xi)$. Then
  $$\Psi_1 = (x'' - y) \cdot (\Xi - \pi_\sigma \Xi) = (x'' - y + \pi_\sigma y) \cdot \Xi,$$
  $$\phi'_\xi = \nabla_\Xi \Psi_1|_{\Xi=\xi} = x'' - (y - \pi_\sigma y).$$
- Non-degeneracy $\operatorname{rank} d\phi'_\xi \equiv n - d$:
  $\phi''_{\xi y} = -I_n + \pi_\sigma$ implies $\operatorname{rank} \phi''_{\xi y} \equiv n - d$ and $\operatorname{rank} d\phi'_\xi \equiv n - d$.
- non-vanishing $d\phi \neq 0$ for $\xi \neq 0$:
  Since $\phi'_y = \xi \neq 0$ for $\xi \neq 0$, $d\phi \neq 0$ for $\xi \neq 0$.

# 3-5. $\Lambda'_\phi$

Let us express $\sigma \in G_{d,n}$ by $\sigma = \text{span}\langle \omega_1, \dots, \omega_d \rangle$. By using the similar arguments in the previous page, we have at the critical points $x'' = y - \pi_\sigma y$

$$\left(\sigma, x'', y; \phi'_{\sigma,x''}, -\phi'_y\right) = \left(\sigma, x'', y; (y \cdot \omega_1, \dots, y \cdot \omega_d, 1)\xi, \xi\right).$$

By using a rotation on $\sigma$, we can choose $\{\omega_1, \dots, \omega_d\}$ such that

$$y \cdot \omega_1 = \pm|\pi_\sigma y|, \quad y \cdot \omega_2 = \cdots = y \cdot \omega_d = 0.$$

We have at the critical points $x'' = y - \pi_\sigma y$

$$\left(\sigma, x'', y; \phi'_{\sigma,x''}, -\phi'_y\right) = \left(\sigma, x'', y; (\pm|\pi_\sigma y|, 0, \dots, 0, 1)\xi, \xi\right).$$

This completes the proof of Theorem 9. □

# 4-1. Main Theorem

- Suppose that the metal region $D \subset \mathbb{R}^n$ is a disjoint union of $D_j$ ($j = 1\dots, J$) which are simply connected, strictly convex and bounded with smooth boundaries $\Sigma_j := \partial D_j$. Set $\Sigma := \partial D$.
- For $j \neq k$ $\mathscr{L}_{jk}$ denotes the set of hyperplanes tangential to $\Sigma_j$ and $\Sigma_k$. Set $\mathscr{L} := \cup \mathscr{L}_{jk}$.
- $\chi_{D_j} \in I^{-1+1/2-n/4}(N^*\Sigma_j)$, $\chi_D \in I^{-1+1/2-n/4}(N^*\Sigma)$ (c.f. $f(x.y)$).

**Theorem 10**

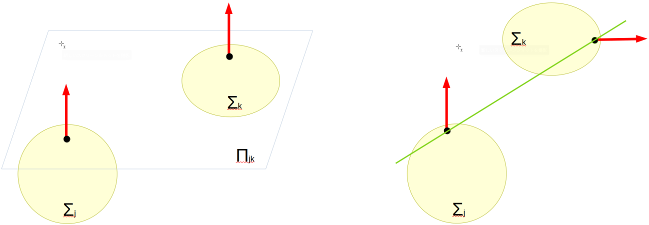
*Away from $\Sigma$, the nonlinear part of the CT image*

$$f_{MA} := f_{CT} - f_{E_0} = \sum_{k=1}^{\infty} A_k (a\varepsilon)^{2k} \mathcal{R}_d^* (-\Delta_{x''})^{d/2} [(\mathcal{R}_d \chi_D)^{2k}]$$

*belongs to $I^{-(d+3)/2+1/2-n/4}(N^*\mathscr{L})$, and its principal symbol does not vanish. This means that $f_{MA}$ causes the metal streaking artifacts.*

# 4-2. Illustration of Theorem 10

- Park-Choi-Seo (CPAM, 2017) proved that $\mathbf{WF}(\boldsymbol{F_{MA}}) = \boldsymbol{N}^*\mathscr{L}$ for $(\boldsymbol{n}, \boldsymbol{d}) = (\mathbf{2}, \mathbf{1})$.
- Theorem 10 is the generalization of Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) for $(\boldsymbol{n}, \boldsymbol{d}) = (\mathbf{2}, \mathbf{1})$.
- If $\Sigma_j$ and $\Sigma_k$ have a common tangent hyperplane $\Pi_{jk}$, then their microlocal singularities, which are in the same conormal direction, spread all over $\Pi_{jk}$.
- If $\Sigma_j$ and $\Sigma_k$ have a common tangent $\boldsymbol{d}$-plane with $\boldsymbol{d} \leqq \boldsymbol{n} - \mathbf{2}$ and the directions of their microlocal singularities are different, then no nonlinear effect occurs.



# 4-3. The canonical transform of $D$

We need to consider

$$(\mathcal{R}_d\chi_D)^2 = \sum_{j=1}^{J}(\mathcal{R}_d\chi_{D_j})^2 + 2\sum_{1\leqq j<k\leqq J}\mathcal{R}_d\chi_{D_j}\cdot\mathcal{R}_d\chi_{D_k},$$

$$\begin{aligned}\Lambda'_\phi\circ N^*\Sigma_j = \{&(\sigma, y-\pi_\sigma y, y; (\pm|\pi_\sigma y|, 0, \dots, 0, 1)\eta) : \\ &(y,\eta)\in N^*\Sigma_j\setminus 0, \sigma\in G_{d,n}, \sigma\subset\eta^\perp\}.\end{aligned}$$

Set

$$\begin{aligned}S_j &:= \pi_{G(d,n)}(\Lambda'_\phi\circ N^*\Sigma_j) \\ &= \{(\sigma, y-\pi_\sigma y) : y\in\Sigma_j\setminus 0, \sigma\in G_{d,n}, \sigma\subset T_y\Sigma_j\}.\end{aligned}$$

## Lemma 11

- codim $S_j = 1$, *and* $N^*S_j = \Lambda'_\phi\circ N^*\Sigma_j$.
- *If* $j\neq k$ *and* $S_j\cap S_k\neq\emptyset$, *then* $S_j$ *intersects* $S_k$ *transversally.*

# 4-4. $S_j \cap S_k$ and its conormal bundle

If $(\sigma, x'') \in S_j \cap S_k$, then there exist $y_j \in \Sigma_j$ and $y_k \in \Sigma_k$ such that $\sigma \subset T_{y_j}\Sigma_j \cap T_{y_k}\Sigma_k$ and $x'' = y_j - \pi_\sigma y_j = y_k - \pi_\sigma y_k$. Using the same notation we set for $l = d, \dots, n-1$

$$(S_j \cap S_k)_l := \{(\sigma, x'') \in S_j \cap S_k : \dim(T_{y_j}\Sigma_j \cap T_{y_k}\Sigma_k) = l\}.$$

**Lemma 12**

- *If $N^*_{y_j}\Sigma_j = N^*_{y_k}\Sigma_k$, then $T_{y_j}\Sigma_j = T_{y_k}\Sigma_k$ and $l = n-1$.*
- $\operatorname{codim}(S_j \cap S_k)_l = d(n-l-1) + 2$.

Set $S_{jk} := (S_j \cap S_k)_{n-1}$. Then $\operatorname{codim} S_{jk} = 2$, and

$$N^* S_{jk} = \{ (\sigma, x''; (t, 0, \dots, 0, 1)\eta) : (\sigma, x'') \in S_{jk}, t \in \mathbb{R}, \eta \in N^*_{y_j}\Sigma_j \setminus 0 \},$$

$$(t, 0, \dots, 0, 1)\eta = \frac{t_k - t}{t_k - t_j}(t_j, 0, \dots, 0, 1)\eta + \frac{t - t_j}{t_k - t_j}(t_k, 0, \dots, 0, 1)\eta.$$

# 4-5. Interaction near $(S_j \cap S_k)_l$ ($l < n-1$)

Theorem 9 implies that $\mathcal{R}_d \chi_{D_j} \in I^{-(d+2)/2+1/2-N(d,n)/4}(N^* S_j)$. We show that we can neglect the interaction near $(S_j \cap S_k)_l$ ($l < n-1$). Fix an arbitrary $\Xi \in (S_j \cap S_k)_l$. We can choose local coordinates $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N(d,n)-2}$ such that $\Xi = 0$. $u \in I^{-\mu}(N^* S_j)$ and $v \in I^{-\mu}(N^* S_k)$ are given by

$$u = \int_{\mathbb{R}} e^{ix\xi} a(y, z, \xi) d\xi, \quad v = \int_{\mathbb{R}} e^{iy\eta} b(x, z, \eta) d\eta,$$

and since $\langle \xi \rangle^{-N} \langle \eta \rangle^{-N} \langle D_x \rangle^N \langle D_y \rangle^N e^{i(x\xi+y\eta)} = e^{i(x\xi+y\eta)}$

$$\begin{aligned} uv &= \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} a(y, z, \xi) b(x, z, \eta) d\xi d\eta \\ &= \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} \frac{\langle D_y \rangle^N a(y, z, \xi)}{\langle \xi \rangle^N} \frac{\langle D_x \rangle^N b(x, z, \eta)}{\langle \eta \rangle^N} d\xi d\eta \end{aligned}$$

for any $N > 0$ near $0$.

# 4-6. Paired Lagrangian distributions

## Definition 13 (Paired Lagrangian distributions)

Let $\mu, \nu \in \mathbb{R}$, and let $\Lambda_0, \Lambda_1$ be conic Lagrangian submanifold of $T^*X \setminus 0$. We say that $u \in \mathscr{D}'(X)$ belongs to $I^{\mu,\nu}(\Lambda_0, \Lambda_1)$ if $\mathrm{WF}(u) \subset \Lambda_0 \cup \Lambda_1$ and away from $\Lambda_0 \cap \Lambda_1$

$$u \in I^{\mu+\nu}(\Lambda_0 \setminus \Lambda_1), \quad u \in I^{\mu}(\Lambda_1).$$

## Lemma 14 (Greenleaf-Uhlmann, 1993)

*Let $X$ be an $N$-dimensional manifold, and let $Y, Z$ be submanifolds of $X$ with $\operatorname{codim} Y = k_1$ and $\operatorname{codim} Z = l_1$ respectively. Suppose $Y \pitchfork Z$ and set $\operatorname{codim} Y \cap Z = k_1 + k_2 = l_1 + l_2$. Then*

$$\begin{aligned} & I^{\mu+k_1/2-N/4}(N^*Y) \cdot I^{\nu+k_1/2-N/4}(N^*Z) \\ \subset & I^{\mu+k_1/2-N/4,\nu+k_2/2}(N^*(Y \cap Z), N^*Y) \\ + & I^{\nu+l_1/2-N/4,\mu+l_2/2}(N^*(Y \cap Z), N^*Z). \end{aligned}$$

# 4-7. Outline of Proof of Theorem 10

- Set $\mathscr{A} := \sum_{j \neq k} I^{-(d+2)/2+1/2-N(d,b)/4,-(d+2)/2+1/2}(N^* S_{jk}, N^* S_j)$.
- Note that $I^{-(d+2)/2+1/2-N(d,b)/4}(N^* S_j) \subset \mathscr{A}$.
- Lemma 14 proves that $(\mathcal{R}_d \chi_D)^2 \in \mathscr{A}$.
- It follows that $\mathscr{A}$ is an algebra. In particular

$$P_{MA} := \sum_{k=1}^{\infty} A_k (\alpha\varepsilon)^{2k} (\mathcal{R}_d \chi_D)^{2k} \in \mathscr{A}.$$

- Applying Lemma 15 to $P_{MA}$, we prove Theorem 10.

**Lemma 15**

*$\mathcal{R}_d^*(-\Delta_{x''})^{d/2}$ is a FIO of order $\dfrac{n+N(d,n)}{4} - \dfrac{n-d}{2}$ with a canonical relation $(\Lambda'_\phi)^* := \{(x,y,\xi,\eta) : (y,x,;\eta,\xi) \in \Lambda'_\phi\}$, and $(\Lambda'_\phi)^* \circ N^* S_j = N^* \Sigma_j$, $(\Lambda'_\phi)^* \circ N^* S_{jk} = N^* \mathscr{L}_{jk}$.*

# 4-8. $I^{\mu,\nu}(N^*S_{jk}, N^*S_j)$ is given by oscillatory integrals.

If $u \in I^{-(d+2)/2+1/2-N(d,b)/4,-(d+2)/2+1/2}(N^*S_{jk}, N^*S_j)$, we can choose local coordinates $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N(d,n)-2}$ and find $a(x, y, z, \xi, \eta)$ such that $S_j = \{x = 0\}$, $S_{jk} = \{x = y = 0\}$,

$$\partial^{\gamma}_{x,y,z}\partial^{\alpha}_{\xi}\partial^{\beta}_{\eta}a(x, y, z, \xi, \eta) = \mathcal{O}(\langle\xi;\eta\rangle^{-(d+2)/2-\alpha}\langle\eta\rangle^{-(d+2)/2-\beta}),$$

$$u(x, y, z) = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)}a(x, y, z, \xi, \eta)d\xi d\eta$$

near $(x, y, z) = 0$. Such a class of symbols is denoted by $\mathcal{B}_d$. We shall evaluate the products of such paired Lagrangian distributions. To show that $\mathscr{A}$ is an algebra, we shall check that $U_d$ is closed under convolutions in $\xi$ and $\eta$.

Pick up $\psi(t) \in C^{\infty}(\mathbb{R})$ such that $0 \leqq \psi(t) \leqq 1$, $\psi(t) = 1$ for $|t| \leqq 1/2$, $0 < \psi(t) < 1$ for $1/2 < |t| < 3/4$, $\psi(t) = 0$ for $|t| \geqq 3/4$. Then $1 - \psi(t) = 1$ for $|t| \geqq 3/4$, $0 < 1 - \psi(t) < 1$ for $1/2 < |t| < 3/4$, $1 - \psi(t) = 0$ for $|t| \leqq 1/2$.

## 4-9. $\left(I^{\mu,\nu}(N^*S_{jk}, N^*S_j)\right)^2 \subset \mathscr{A}$

Suppose that there exists $b(x,y,z,\xi,\eta) \in U_d$ such that

$$v(x,y,z) = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} b(x,y,z,\xi,\eta)\, d\xi d\eta.$$

Then

$$uv = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} c_1(x,y,z,\xi,\eta)\, d\xi d\eta,$$

$$c = \iint_{\mathbb{R}^2} a(x,y,z,\xi-\zeta,\eta-\lambda) b(x,y,\zeta,\lambda)\, d\zeta d\lambda.$$

It suffices to show that $c_1 \in U_d$.

Set $\Phi(\xi,\eta,\zeta,\lambda) := \psi\left(\dfrac{\langle \xi-\zeta,\eta-\lambda\rangle}{\langle \xi;\eta\rangle}\right)$, $\Psi(\eta,\lambda) := \psi\left(\dfrac{\langle \eta-\lambda\rangle}{\langle \eta\rangle}\right)$.

- $|(\xi,\eta)| \sim |(\zeta,\lambda)|$ for $\Phi > 0$.
- $\langle \xi-\zeta,\eta-\lambda\rangle \geqq \langle \xi;\eta\rangle/2$ for $1-\Phi > 0$.
- $\partial_\zeta^\alpha \partial_\lambda^\beta \Phi = \mathcal{O}(\langle \xi-\zeta,\eta-\lambda\rangle^{-\alpha-\beta})$.

## 4-10. $c_1 \in U_d$

Note that $\partial_\xi\bigl(g(\xi-\zeta)\bigr) = -\partial_\zeta\bigl(g(\xi-\zeta)\bigr)$. Using

$$1 = \Phi\Psi + (1-\Phi)\Psi + \Phi(1-\Psi) + (1-\Phi)(1-\Psi),$$

we deduce that

$$\partial_{x,y,z}^{\gamma}\partial_\xi^\alpha\partial_\eta^\beta c = \sum_{\gamma' \leqq \gamma} C(\gamma') \iint_{\mathbb{R}^2} a_{\gamma'}^{\alpha,\beta} b_{\gamma-\gamma'} d\zeta d\lambda$$

$$\begin{aligned} \iint_{\mathbb{R}^2} a_{\gamma'}^{\alpha,\beta} b_{\gamma-\gamma'} d\zeta d\lambda &= \iint_{\mathbb{R}^2} a_{\gamma'} \partial_\zeta^\alpha \partial_\lambda^\beta (\Phi\Psi b_{\gamma-\gamma'}) d\zeta d\lambda \\ &+ \iint_{\mathbb{R}^2} a_{\gamma'}^{\alpha,0} \partial_\lambda^\beta ((1-\Phi)\Psi b_{\gamma-\gamma'}) d\zeta d\lambda \\ &+ \iint_{\mathbb{R}^2} a_{\gamma'}^{0,\beta} \partial_\zeta^\alpha (\Phi(1-\Psi) b_{\gamma-\gamma'}) d\zeta d\lambda \\ &+ \iint_{\mathbb{R}^2} a_{\gamma'}^{\alpha,\beta} (1-\Phi)(1-\Psi) b_{\gamma-\gamma'} d\zeta d\lambda \\ &= \mathcal{O}(\langle \xi;\eta\rangle^{-(d+2)/2-\alpha} \langle \eta\rangle^{-(d+2)/2-\beta}). \end{aligned}$$

## 4-11. $I^{\mu,\nu}(N^*S_{jk}, N^*S_j) \cdot I^{\mu,\nu}(N^*S_{jk}, N^*S_k) \subset \mathscr{A}$

Suppose that there exists $b(x,y,z,\eta,\xi) \in U_d$ such that

$$v(x,y,z) = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} b(x,y,z,\eta,\xi)\, d\xi d\eta.$$

Then

$$uv = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)} p(x,y,z,\xi,\eta)\, d\xi d\eta,$$

$$p = \iint_{\mathbb{R}^2} a(x,y,z,\xi-\zeta,\eta-\lambda) b(x,y,\lambda,\zeta)\, d\zeta d\lambda.$$

In the same way as in the previous page we have

$$\partial_{x,y,z}\partial_\xi{}^\alpha \partial_\eta^\beta p = \mathcal{O}(\langle\xi\rangle^{-(d+2)/2-\alpha}\langle\eta\rangle^{-(d+2)/2-\beta}).$$

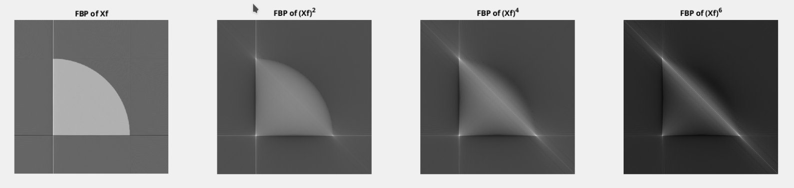
If we set

$$q(x,y,z,\xi,\eta) := \psi\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right) p, \quad r(x,y,z,\eta,\xi) := p - q,$$

then $q, r \in U_d$, which implies $I \cdot I' \subset I + I'$. □

# 5. Discussions

- $P_{MA} \notin \mathcal{R}_d(\mathscr{E}'(\mathbb{R}^n))$.
- Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) studied the case that $D_1, \dots, D_J$ are convex polygons in $\mathbb{R}^2$.




When $D_1, \dots, D_J$ are convex polyhedra in $\mathbb{R}^3$, the situation is more complicated.

- Theorem 9 might be useful to studying limited data problem.
- It might be worth studying these topics in the setting of some kind of Riemannian manifolds.

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