

Kleshchev's decomposition numbers for cyclotomic Hecke algebras

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1 Motivation

Fix \mathbb{F} a field of characteristic $p \geq 0$ throughout and $e \in \{3, 4, \dots\} \cup \{\infty\}$. Let $q \in \mathbb{F}$ be a primitive e th root of unity, (if $e = \infty$ then q is not a root of unity) and $\kappa = (\kappa_1, \dots, \kappa_l) \in (\mathbb{Z}/e\mathbb{Z})^l$. We denote by $\mathcal{H}_n = \mathcal{H}_n(q, \kappa)$ the *cyclotomic Hecke algebra* (of type $G(l, 1, n)$) of degree n with parameters q and κ . It is a deformation of the group algebra of the complex reflection group $G(l, 1, n) = (\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$.

Brundan and Kleshchev [2] have shown that \mathcal{H}_n is a \mathbb{Z} -graded algebra, and Hu and Mathas [3] have shown that it is in fact a *graded cellular* algebra. The cellular structure agrees with that of the Dipper–James–Mathas construction of the cyclotomic Hecke algebra. The cell modules are indexed by the set \mathcal{P}_n^l of l -multipartitions of n and the simple modules by a certain subset $\Theta \subset \mathcal{P}_n^l$ of these.

Ariki's theorem tells us that in fact there are many different parameterisations $\Theta \subset \mathcal{P}_n^l$ for the simple modules, and the Dipper–James–Mathas setup sees only one of these. We would like different cellular structures for each one of these.

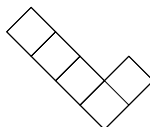
Aim Study graded decomposition numbers for \mathcal{H}_n corresponding to various parameterisations of the simple modules.

In order to do this, we must lift to the setting of quasi-hereditary covers of \mathcal{H}_n . We will see that the diagrammatic Cherednik algebra depends on a weighting, θ , which in turn determines which parameterisation Θ we are seeing. (For any θ we get a quasi-hereditary cover of \mathcal{H}_n , and different covers may yield different parameterisations of simples.)

2 Nested sign sequences

We begin by discussing some level 1 results for graded decomposition numbers which we would like to generalise. Fix $e \in \{2, 3, \dots\} \cup \{\infty\}$ and $i \in \mathbb{Z}/e\mathbb{Z}$ throughout. Note that for this section alone we allow $e = 2$.

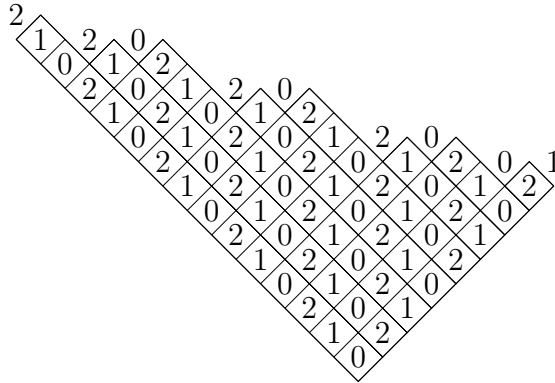
We will take a mirrored-Russian convention for drawing our Young diagrams. For example, the Young diagram for the partition $(4, 1)$ is drawn as



Definition 2.1. Given μ a partition of n , we draw its Young diagram and fill with e -residues. Reading from left to right, we draw a path which we call the *terrain of μ* using the following steps:

$$\begin{cases} / & \text{for each removable } i\text{-node of } [\mu]; \\ \backslash & \text{for each addable } i\text{-node of } [\mu]. \end{cases}$$

Example. Take $e = 3$, $i = 2$ and $\mu = (14, 12, 11, 8, 7, 4, 3, 1)$. The Young diagram $[\mu]$ with residues is



and so the terrain of μ is



Given λ and μ partitions of n with $\lambda \triangleright \mu$ which differ by moving i -nodes, we give our terrain of μ a λ -decoration as follows. To each addable i -node (down steps) of μ which has been added to obtain λ , we associate a “(”, and to each removable i -node (up steps) of μ which is removed to obtain λ we associate a “)”. This gives us a system of nested parentheses, which we identify with the edge they are associated to. We denote the λ -decorated terrain of μ by $P(\mu, \lambda)$.

Example. Continuing with the previous example, if we let $\lambda = (15, 13, 10, 9, 7, 4, 2)$ then the λ -decorated terrain of μ is



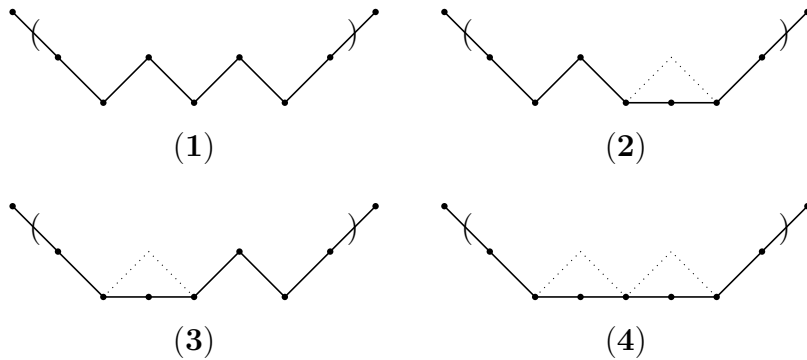
We may order pairs of parentheses by inclusion in the obvious manner; we let $\mathcal{Q}(\mu, \lambda)$ denote the partially ordered set of parentheses on $P(\mu, \lambda)$.

Definition 2.2. Given a pair of parentheses $P \in \mathcal{Q}(\mu, \lambda)$, the set of *lattice paths on $P(\mu, \lambda)$ with respect to P* is the set of all possible paths obtained by replacing some number of ridges formed of edges *strictly* between the parentheses to obtain flattened ridges.

Given two such paths ρ and ρ' , we write $\rho \leq \rho'$ if every vertex in ρ' is at least as high as the corresponding vertex in ρ .

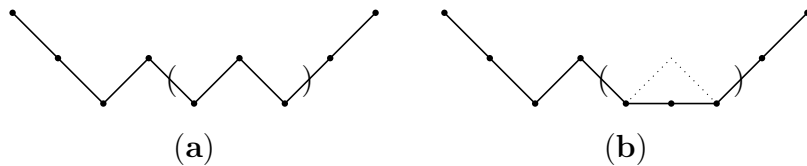
If ρ is a lattice path, we say that ρ has norm $\|\rho\|$ given by the number of non-flattened steps between the parentheses comprising P , plus 1.

Example. Continuing with the previous example, there are four latticed paths with respect to the pair of parentheses on the first and eighth edges of the path, with norms 7, 5, 5 and 3.



The only latticed path with respect to the pair of parentheses on the second and third edge is $P(\mu, \lambda)$ itself, with norm 1.

There are two latticed paths on $P(\mu, \lambda)$ with respect to the pair of parentheses on the fourth and seventh edges of the path, with norms 3 and 1.



Definition 2.3. A *well-nested latticed path* for $P(\mu, \lambda)$ is a collection $\{\rho_P \mid P \in \mathcal{Q}(\mu, \lambda)\}$ of latticed paths such that if $P, Q \in \mathcal{Q}(\mu, \lambda)$ and $P \subset Q$, then $\rho_P \geq \rho_Q$. We let $\Omega(\mu, \lambda)$ denote the set of all well-nested latticed paths. The norm of a well-nested latticed path is given by the sum of the norms of the constituent paths.

Example. Continuing with our example, all triples are well-nested latticed paths except $((1), P(\mu, \lambda), (b))$ and $((3), P(\mu, \lambda), (b))$.

tt **Theorem 2.4** [5, Theorem 4.4]. *Suppose $\lambda \triangleright \mu$ are partitions of n which differ only by moving i -nodes. Then the graded decomposition number $d_{\lambda\mu}(v)$ for the Hecke algebra of type A is independent of the characteristic of \mathbb{F} and is given by*

$$d_{\lambda\mu}(v) = \sum_{\omega \in \Omega(\mu, \lambda)} v^{|\omega|}.$$

Example. Continuing with our example, we have $d_{\lambda\mu}(v) = v^{11} + 2v^9 + 2v^7 + v^5$.

3 The diagrammatic Cherednik algebra

We will now discuss the combinatorics of Webster's diagrammatic Cherednik algebra [6].

Given a *weighting* $\theta \in \mathbb{R}^l$, we have an associated diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ which is a quasi-hereditary cover of the cyclotomic Hecke algebra \mathcal{H}_n . The algebra $A(n, \theta, \kappa)$ is defined using a diagram calculus similar to that of Khovanov and Lauda [4], but more involved. We omit the full definition due to its complexity, but rather give a flavour of the underlying combinatorics. Note that while the algebra is difficult to work with, this difficulty pays off in terms of yielding representation theoretic results!

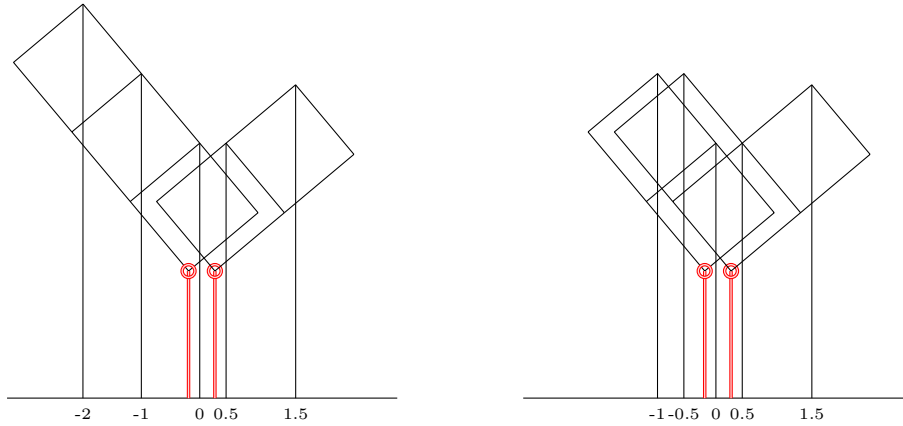
Definition 3.1. Given a weighting θ and some $\lambda \in \mathcal{P}_n^l$, we draw the Young diagram $[\lambda]$ of λ by placing the first node of the j th component at point θ_j on the x -axis, with all boxes having diagonals of length 2. We tilt our Young diagrams ever so slightly clockwise, so that the top corners of different nodes have different x -coordinates.

The *loading* i_λ is the n -tuple of real numbers given by projecting the top vertices of boxes of $[\lambda]$ onto the real line, along with the residue associated to each box.

Definition 3.2. We write $\lambda \triangleright_\theta \mu$, and say λ θ -dominates μ , if for every real number a and every $j \in \mathbb{Z}/e\mathbb{Z}$, there are at least as many j -nodes to the left of a in i_λ as in i_μ .

If $l = 1$ this order is a coarsening of the usual dominance order. Similarly, if θ is *well-separated* (that is, if $\theta_1 \ll \theta_2 \ll \dots \ll \theta_l$), this order is a coarsening of the usual dominance order on multipartitions. In general, the θ -dominance order depends subtly on θ .

Example. Suppose $\theta = (0, 0.5)$. If $\lambda = ((3), (1^2))$ and $\mu = ((2), (2, 1))$, then we draw $[\lambda]$ and $[\mu]$ as below, with the loadings given by projections onto the real line.



Definition 3.3. Let $\lambda, \mu \in \mathcal{P}_n^l$. A semistandard tableau of shape λ and weight μ is a map $T : [\lambda] \rightarrow i_\mu$ which respects residues and for all admissible r, c, k ,

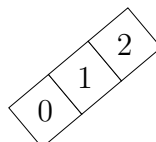
- $T(1, 1, k) \geq \theta_k$,
- $T(r, c, k) \geq T(r - 1, c, k) + 1$,
- $T(r, c, k) \geq T(r, c - 1, k) - 1$.

We denote the set of semistandard tableaux of shape λ and weight μ by $\text{SStd}(\lambda, \mu)$.

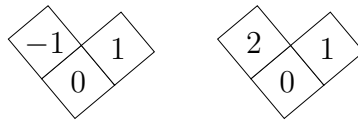
Note that $\text{SStd}(\lambda, \mu) = \emptyset$ unless $\lambda \triangleright_\theta \mu$.

Example. Let $e = 3$, $l = 1$, $n = 3$ and $\theta = \kappa = (0)$.

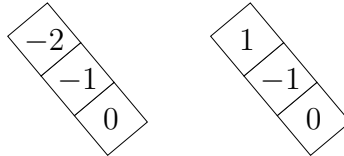
The only semistandard tableau of shape (1^3) is



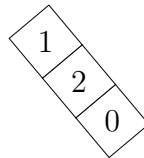
There is one semistandard tableau in $\text{SStd}((2, 1), (2, 1))$ and one in $\text{SStd}((2, 1), (1^3))$.



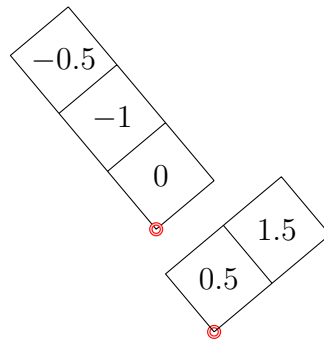
There is one semistandard tableau in $\text{SStd}((3), (3))$, and one in $\text{SStd}((3), (2, 1))$.



Technically, the coordinates written in the boxes should take into account the slight tilting of our Young diagrams. We have omitted them to keep the diagrams neat, but they account for the reason the following tableau is **not** semistandard.



Example. Suppose $e = 3$, $\theta = (0, 0.5)$ and $\kappa = (0, 1)$. With λ and μ as before, the only element of $\text{SStd}(\lambda, \mu)$ is



Theorem 3.4 [6, Theorem 4.10]. *The diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ is a graded cellular algebra with respect to the θ -dominance order and a basis indexed by $\text{SStd}(\lambda, \mu)$ as λ and μ range over \mathcal{P}_n^l .*

In particular we have graded standard modules $\Delta(\lambda) = \langle C_{\mathbf{T}} \mid \mathbf{T} \in \text{SStd}(\lambda, -) \rangle_{\mathbb{F}}$ with graded simple heads $L(\lambda)$ forming a complete set of graded simple modules, up to grading shift.

Over \mathbb{C} , the module category of $A(n, \theta, \kappa)$ is equivalent to category \mathcal{O} for the rational cyclotomic Cherednik algebra. If θ is well-separated (that is, $\theta_j - \theta_k \gg 0$ for all j and k), then $A(n, \theta, \kappa)$ is Morita equivalent to the q -Schur algebra of Dipper–James–Mathas over arbitrary fields.

We would like to compute the graded decomposition numbers $d_{\lambda\mu}(v)$ for $A(n, \theta, \kappa)$.

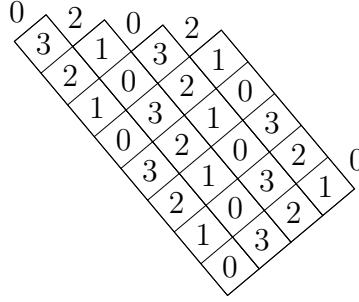
Example. With the level 1 example for $e = n = 3$, the graded decomposition numbers may be obtained from the semistandard tableaux, almost for free.

4 Subquotients of $A(n, \theta, \kappa)$

Pick a set $S \subset \mathbb{Z}/e\mathbb{Z}$ of residues which is adjacency-free; that is, if $i \in S$ then $i \pm 1 \notin S$.

Suppose $\gamma \in \mathcal{P}_n^l$ has no removable i -nodes for any $i \in S$, and let \mathcal{M} denote a multiset of residues in S . Now let $\Gamma_{\mathcal{M}}$ denote the set of multipartitions obtained from γ by adding nodes of residues in \mathcal{M} .

Example. Let $e = 4$, $S = \{0, 2\}$ and $\gamma = (8, 7, 6, 5)$. If $\mathcal{M} = \{0, 0, 2\}$ then we could have for instance, $\lambda = (9, 8, 7, 5)$, $\mu = (8, 7^2, 6, 1) \in \Gamma_{\mathcal{M}}$.



Fact: The set $\Gamma_{\mathcal{M}}$ is an interval in the θ -dominance order, with maximal element given by placing all nodes as far left as possible, and minimal element given by placing all nodes as far right as possible. Thus we may take a subquotient $A_{\Gamma_{\mathcal{M}}}$ whose standard modules are indexed by all $\lambda \in \Gamma_{\mathcal{M}}$. In particular, for $\lambda, \mu \in \Gamma_{\mathcal{M}}$, the graded decomposition number $d_{\lambda\mu}(v)$ for $A_{\Gamma_{\mathcal{M}}}$ is the same as the corresponding decomposition number for $A(n, \theta, \kappa)$.

Theorem 4.1 [1, Theorem 3.8]. *If \mathcal{M} is adjacency-free and $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_{e-1}$ is a decomposition of \mathcal{M} into disjoint residues, then*

$$A_{\Gamma_{\mathcal{M}}} \cong A_{\Gamma_{\mathcal{M}_0}} \otimes A_{\Gamma_{\mathcal{M}_1}} \otimes \dots \otimes A_{\Gamma_{\mathcal{M}_{e-1}}}$$

as graded \mathbb{F} -algebras.

Thus, we may from now on consider the subquotient A_{Γ_m} , corresponding to partitions obtained from some γ by adding m i -nodes for some fixed residue i .

Example. Continuing with the previous example, we have that

$$d_{\lambda\mu}(v) = d_{(9,7^2,5)(8,7^2,5,1)}(v) \times d_{(8^2,6,5)(8,7,6^2)}(v).$$

5 Isomorphisms and decomposition numbers

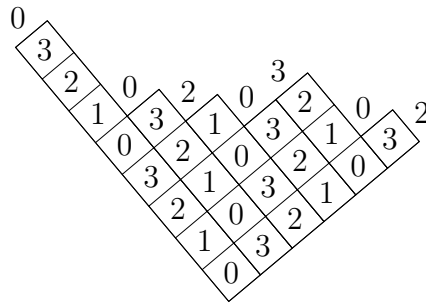
Theorem 5.1 [1, Proposition 4.11 & Theorem 4.12]. *Let $e, \bar{e} \in \{3, 4, \dots\} \cup \{\infty\}$ with $i \in \mathbb{Z}/e\mathbb{Z}$ and $\bar{i} \in \mathbb{Z}/\bar{e}\mathbb{Z}$, and suppose γ (respectively $\bar{\gamma}$) is a multipartition with no removable i -nodes (respectively \bar{i} -nodes) and x addable i -nodes (respectively \bar{i} -nodes) for some x . Then the corresponding subquotients A_{Γ_m} and $A_{\bar{\Gamma}_m}$ are isomorphic as graded vector space over \mathbb{F} .*

Moreover, if $\mathbb{F} = \mathbb{C}$, we have

$$d_{\lambda\mu}(v) = d_{\bar{\lambda}\bar{\mu}}(v)$$

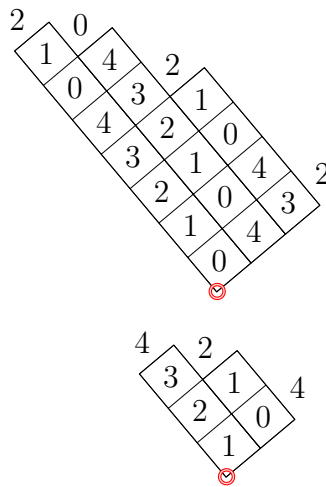
for $\lambda, \mu \in \Gamma_m$ and $\bar{\lambda}, \bar{\mu} \in \bar{\Gamma}_m$.

Example. Let $e = 4$ and $\bar{e} = 5$. Take $\gamma = (8, 5, 4, 3^2, 1)$.



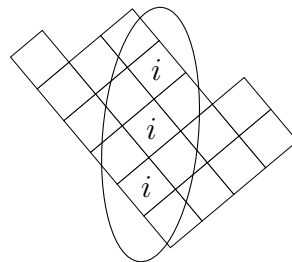
Then γ has no removable 0-nodes and four addable 0-nodes.

Let $\bar{\kappa} = (0, 1)$, $\bar{\theta} = (0, 0.5)$ and $\bar{\gamma} = ((7, 6, 4), (3, 2))$.



Then $\bar{\gamma}$ has no removable 2-nodes and four addable 2-nodes. Thus we can compare decomposition numbers in A_{Γ_m} with those in $A_{\bar{\Gamma}_m}$, where Γ_m is the set of partitions obtained from γ by adding m 0-nodes, and $\bar{\Gamma}_m$ is the set of bipartitions obtained from $\bar{\gamma}$ by adding m 2-nodes. In particular, we can apply the combinatorics of Theorem 2.4 to this level 2 example and calculate the corresponding decomposition numbers!

Our proof of the graded vector space isomorphism is by an explicit construction, which we verify by examining i -diagonals in $[\gamma]$. Without going into the details, we apply the defining relations of the algebra “close to the i -diagonals” and our proof is based around this idea, using some case analysis. The i -diagonals are essentially just diagonals in the Young diagram which have residue i , as pictured below.

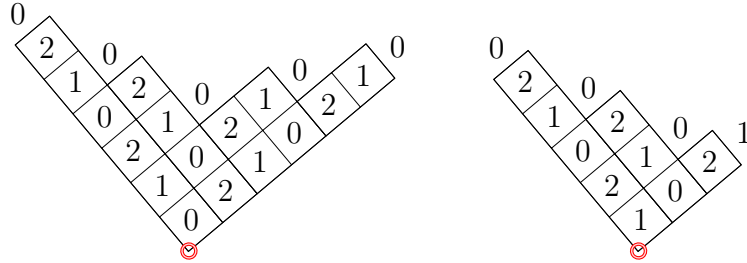


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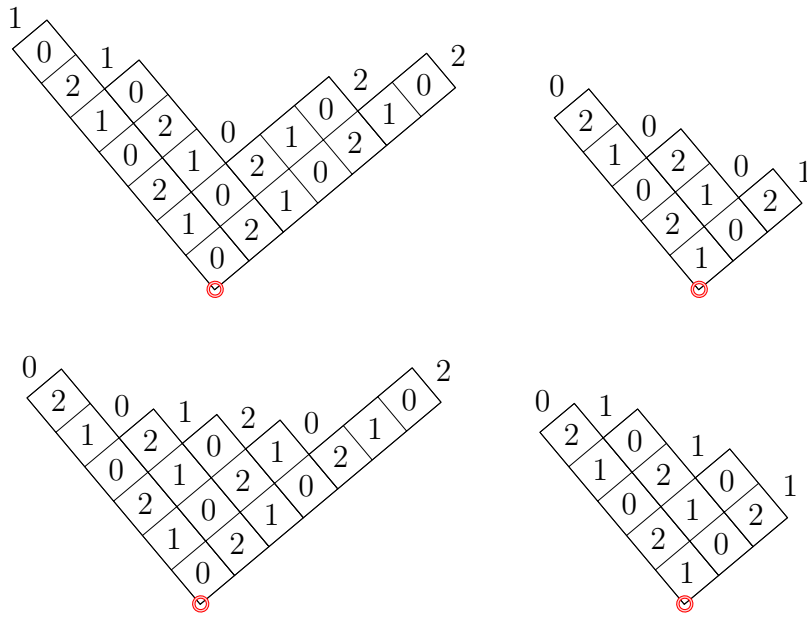
Theorem 5.2 [1, Theorems 4.28 & 4.30]. Take $e, \bar{e}, i, \bar{i}, \gamma, \bar{\gamma}$ as before. Under some extra conditions on the i -diagonals of γ and $\bar{\gamma}$, the isomorphism of the previous theorem is an isomorphism of graded algebras.

In particular, if κ contains at most one instance of the residue i , the graded decomposition numbers are parabolic Kazhdan–Lusztig polynomials, and can be calculated explicitly as in Theorem 2.4.

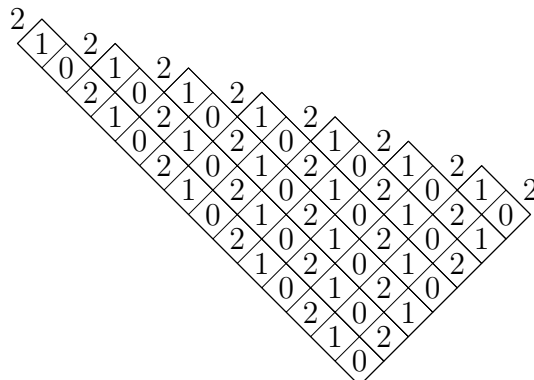
Example. Let $\kappa = (0, 1)$ and $e = 3$, $i = 0$. For $\theta = (0, 20)$, $\gamma = ((6, 4, 2^2, 1^2), (5, 3, 1))$. Then $[\gamma]$ is



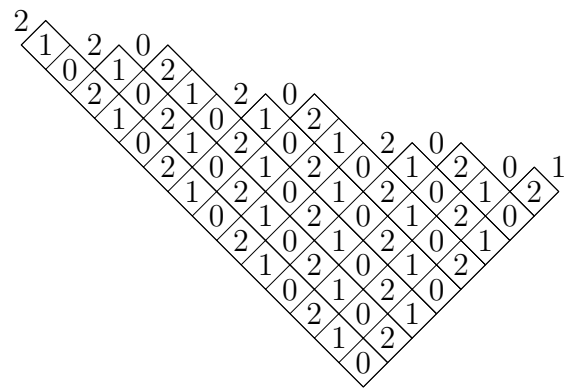
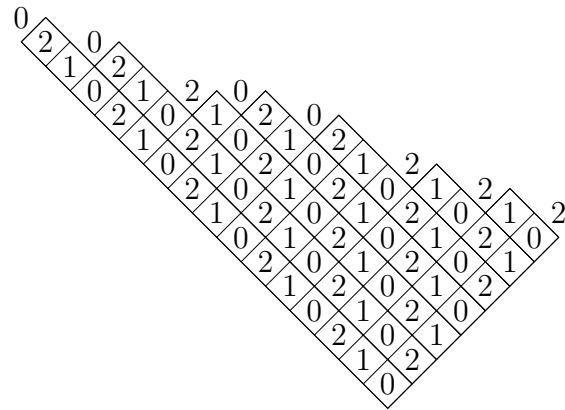
Setting $\lambda = ((7, 5, 2^3, 1^2), (5, 3, 1))$ and $\mu = ((6, 4, 3, 2, 1^3), (5, 4, 2))$, we have the Young diagrams below.



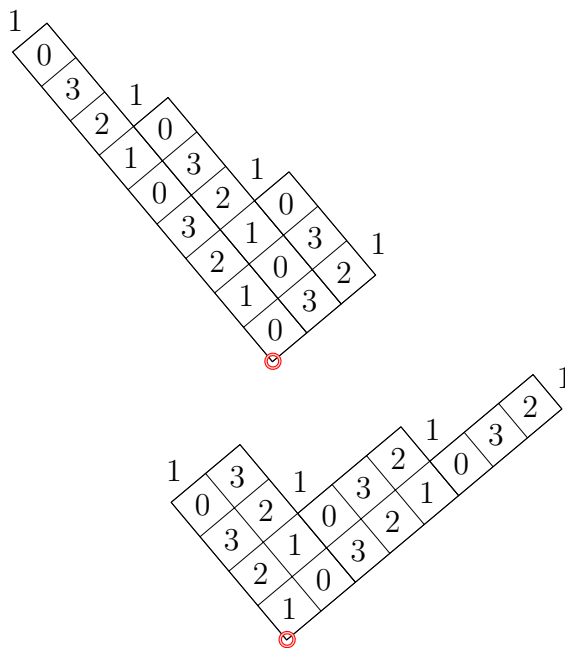
Now take $\bar{e} = 3$, $\bar{i} = 2$ and $\bar{\gamma} = (14, 12, 10, 8, 6, 4, 2)$. The Young diagram $[\bar{\gamma}]$ with residues is



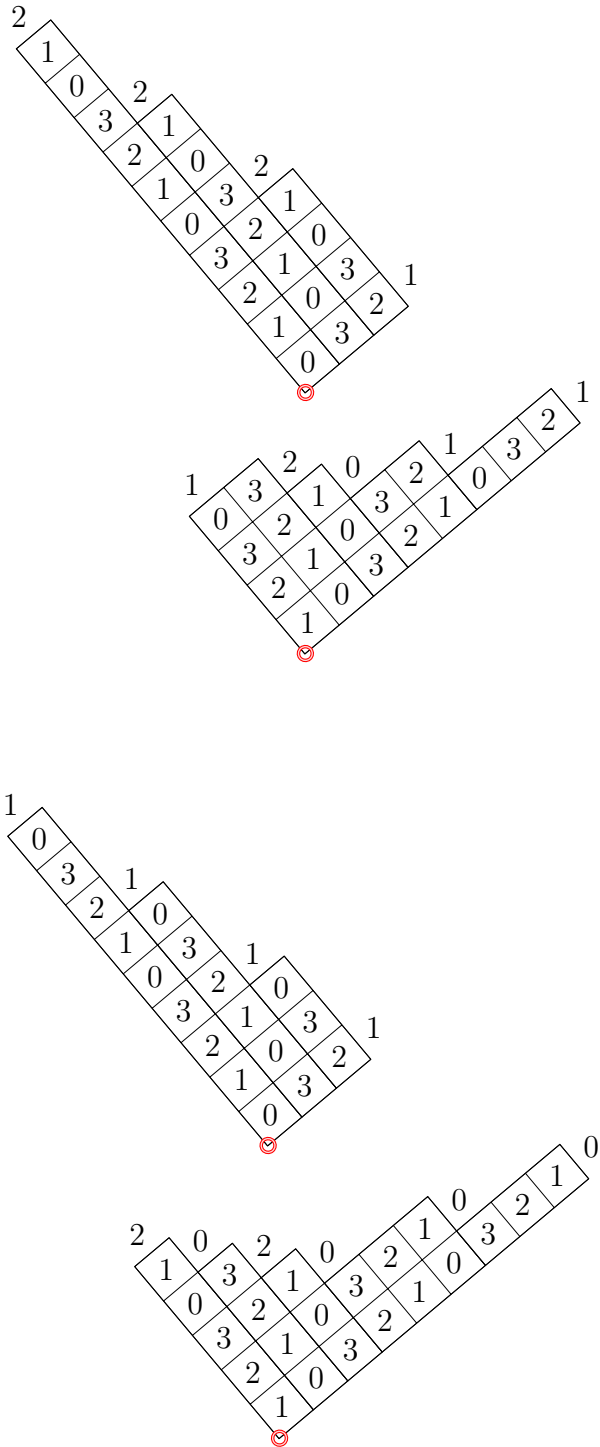
Setting $\bar{\lambda} = (15, 13, 10, 9, 7, 4, 2)$ and $\bar{\mu} = (14, 12, 11, 8, 7, 4, 3, 1)$, we have the Young diagrams below.



Finally, take $\bar{e} = 4$, $\bar{i} = 1$, $\bar{\theta} = (0, 0.5)$, $\bar{\gamma} = ((9, 6, 3), (4^2, 2^3, 1^3))$. Then $[\bar{\gamma}]$ is



Taking $\bar{\lambda} = ((10, 7, 4), (4^2, 3, 2^2, 1^3))$ and $\bar{\mu} = ((9, 6, 3), (5, 4, 3, 2^3, 1^3))$, we get the Young diagrams below.



All three examples involve the same sequences of addable and removable nodes (and thus the same decorated path), and we can compute that $d_{\lambda\mu}(v) = v^{11} + 2v^9 + 2v^7 + v^5$ in all three cases.

As it happens, our condition on the i -diagonals in Theorem 5.2 is not satisfied by these examples, and thus we do not get isomorphisms between the three blocks in positive characteristic. In light of the decomposition numbers matching up, we expect that they are Morita equivalent to each other!

References

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