Strictly wild blocks of type A Hecke algebras

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Hecke algebras

Fix $\mathbb{F}$: an alg. closed field of char. $p \geqslant 0$ throughout.
The Iwahori–Hecke algebra of the symmetric group is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i - q)(T_i + 1) &= 0 && \text{for all } i,\\
T_i T_j &= T_j T_i && \text{for } |i-j| > 1,\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2,
\end{aligned}$$

where $q \in \mathbb{F}$ is a primitive eth root of unity.
$\mathscr{H}_n$ is semisimple if $e > n$.

The *Specht modules* $\{\mathrm{S}^\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions λ of n.

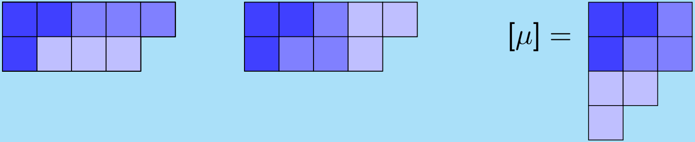
If $e \leqslant n$, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } e\text{-regular}\}$.

Blocks

Specht modules S^λ and S^μ (or simple modules D^λ and D^μ) are in the same block of $\mathscr{H}_n$ if and only if λ and μ *have the same core*.

Example

Let $\lambda = (5, 4)$, $\mu = (3^2, 2, 1)$, and $e = 3$. Then λ and μ are in the same block:

$[\lambda] =$  $[\mu] =$

The *weight* of a partition is the number of e-rim hooks that can be removed before obtaining the core. e.g. $w = 3$ above, with core the empty partition.

Representation type

Definition

The representation type of an $\mathbb{F}$-algebra A is said to be:

- *finite* if it has finitely many indecomp. modules, up to isom.;
- *tame* if for any d, all but fin. many d-dimensional indecomp. modules lie in fin. many one-parameter families, up to isom.;
- *wild* if $\exists$ a fin.-gen. A–$\mathbb{F}\langle X, Y\rangle$-bimodule M, which is free as a right $\mathbb{F}\langle X, Y\rangle$-module, s.t. the functor $M \otimes_{\mathbb{F}\langle X,Y\rangle} - : \mathbb{F}\langle X, Y\rangle\text{-mod} \to A\text{-mod}$ preserves indecomposability and isomorphism classes.

Theorem (Drozd, 1979)

Any $\mathbb{F}$-algebra A has representation type that is exactly one of the above three types.

Representation type of blocks of $\mathscr{H}_n$

Theorem (Erdmann–Nakano, 2002)

Let $e \geqslant 3$, and let B be a weight w block of a Hecke algebra $\mathscr{H}_{\mathbf{n}}$. Then B is

- *simple if $w = 0$;*
- *of finite representation type if $w = 1$;*
- *wild if $w \geqslant 2$.*

Strictly wild algebras

Definition

An $\mathbb{F}$-algebra A is said to be:

- *wild* if $\exists$ a fin.-gen. A–$\mathbb{F}\langle X, Y\rangle$-bimodule M, which is free as a right $\mathbb{F}\langle X, Y\rangle$-module, s.t. the functor $M \otimes_{\mathbb{F}\langle X,Y\rangle} - : \mathbb{F}\langle X, Y\rangle\text{-mod} \to A\text{-mod}$ preserves indecomposability and isomorphism classes.
- *strictly wild* if the functor above is *full*.

Not every wild algebra is strictly wild. e.g. $\mathbb{F}[x, y, z]/(x, y, z)^2$ is wild, but not strictly wild.

Schurian-finiteness

For any $\mathbb{F}$-algebra A, we say that an A-module M is Schurian (or a *brick*) if $\mathrm{End}_A(M) \cong \mathbb{F}$. We say that A is Schurian-finite (or *brick-finite*) if there are only finitely many isomorphism classes of Schurian A-modules, and Schurian-infinite (or *brick-infinite*) otherwise.

Schurian modules must be indecomposable, so clearly

$$\text{representation-finite} \Rightarrow \text{Schurian-finite}.$$

The converse is not true in general – e.g. preprojective algebras of type other than A_n for $1 \leqslant n \leqslant 4$ are representation-infinite, but Schurian-finite.

Strictly wild vs. Schurian-finite

A result of Demonet, Iyama and Jasso (2019) yields that A is Schurian-finite if and only if it is τ-tilting finite.

Fact

A strictly-wild algebra is Schurian-infinite (brick-infinite).
In fact, stronger still, a strictly-wild algebra is actually *brick-wild*.

Reduction

Proposition

Let Q be a finite, acyclic, connected quiver.

- Q not finite type ADE, $\Rightarrow$ path algebra $\mathbb{F}Q$ is brick-infinite.
- Q not finite/affine type ADE, $\Rightarrow$ $\mathbb{F}Q$ is strictly wild.

Proposition

Let A be a finite-dimensional algebra.

- If the Gabriel quiver of A contains the quiver of an affine Dynkin diagram with zigzag orientation (i.e. every vertex is a sink or a source) as a subquiver, then A is Schurian-infinite.
- If the Gabriel quiver of A contains a subquiver as above + an extra vertex connected to it, then A is strictly wild.

Why does this work?

If the Gabriel quiver of A contains such a quiver Q, then since Q has no paths of length > 1, we have a surjection $A \to \mathbb{F}Q$. This gives us a fully faithful exact functor $\mathbb{F}Q$-mod $\to A$-mod.

This functor sends bricks to bricks, so Schurian-infiniteness is preserved. It also gives us a fully faithful exact functor $\mathbb{F}\langle X, Y \rangle$-mod $\to A$-mod, so strict wildness is preserved, too.

We want to determine which blocks of type A Hecke algebras are strictly wild/Schurian-infinite using this proposition.

Graded decomposition numbers

Results of Brundan, Kleshchev, and Wang $\rightsquigarrow \mathscr{H}_n$ is isomorphic to a cyclotomic KLR algebra, and its Specht modules and simple modules may be graded.

The graded decomposition number $d^{e,p}_{\lambda\mu}(v)$ is defined to be the graded composition multiplicity of D^μ in S^λ. In other words

$$d^{e,p}_{\lambda\mu}(v) = [\mathsf{S}^\lambda : \mathsf{D}^\mu]_v = \sum_{d\in\mathbb{Z}} [\mathsf{S}^\lambda : \mathsf{D}^\mu\langle d\rangle] v^d \in \mathbb{N}[v, v^{-1}].$$

Graded decomposition numbers

Using a result of Shan on Jantzen filtrations and radical filtrations of Weyl modules for q-Schur algebras, we can deduce the following.

Lemma

Suppose that $e \geqslant 3$, $p = 0$, and λ, μ are e-regular partitions of n. If the coefficient of v in $d^{e,0}_{\lambda\mu}(v)$ is nonzero, then

$$\mathrm{Ext}^1(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \mathrm{Ext}^1(\mathrm{D}^\mu, \mathrm{D}^\lambda) \neq 0.$$

Combining this with an argument involving idempotent truncation, we're able to obtain our main tool for showing that a given block of $\mathscr{H}_n$ is Schurian-infinite or strictly wild.

Key Proposition (Ariki–Lyle–S., S.)

Suppose $e \geqslant 3$ & $p \geqslant 0$. If the char 0 graded decomposition matrix has (†) (resp. (‡)) as a submatrix, and $d^{e,p}_{\lambda\mu}(1) = d^{e,0}_{\lambda\mu}(1) \in \{0, 1\}$ for all row labels λ, μ of the submatrix, then the block is Schurian-infinite (resp. strictly wild).

$$\begin{pmatrix} 1 & & & \\ v & 1 & & \\ v & * & 1 & \\ * & v & v & 1 \end{pmatrix} \qquad (\dagger) \qquad\qquad \begin{pmatrix} 1 & & & & \\ v & 1 & & & \\ * & v & 1 & & \\ * & v & * & 1 & \\ * & * & v & v & 1 \end{pmatrix} \qquad (\ddagger)$$

Why these matrices? e-regular partitions $\lambda, \mu, \nu, \omega$ with submatrix (†) $\rightsquigarrow$ if $p = 0$, the previous lemma gives subquiver

$$\begin{tikzcd} \lambda \arrow[r, dash] \arrow[d, dash] & \mu \arrow[d, dash] \\ \nu \arrow[r, dash] & \omega \end{tikzcd}$$

which is $A_3^{(1)}$ $\rightsquigarrow$ the result (in characteristic 0).

e-regular partitions $\kappa, \lambda, \mu, \nu, \omega$ with submatrix ($\ddagger$) $\rightsquigarrow$ if $p = 0$, the previous lemma gives subquiver

$$\begin{array}{ccccc} \kappa & \text{———} & \lambda & \text{———} & \mu \\ & & | & & | \\ & & \nu & \text{———} & \omega \end{array}$$

which is $A_3^{(1)\wedge}$ $\rightsquigarrow$ the result (in characteristic 0).

Main results

(Weight 0 and 1 blocks of $\mathscr{H}_n$ are representation-finite and therefore Schurian-finite.)

Theorem (Ariki–Lyle–S., 2023)

Suppose $e \geqslant 3$, and that B is any block of weight $\geqslant 2$. Then B is Schurian-infinite in any characteristic.

Theorem (S., 2024)

Suppose $e \geqslant 3$, and that B is any block of $\mathscr{H}_n$ with weight $\geqslant 2$. If $e = 3$, suppose further that B is not (Scopes equivalent to) the weight 2 Rouquier block. Then B is strictly wild, and therefore brick-wild, in any characteristic.

Row removal

We will illustrate our method of proof in an example. First, we'll need the following row removal results.

Theorem (Chuang–Miyachi–Tan, 2002)

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$. If $\lambda_1 = \mu_1$, let $\bar{\lambda} = (\lambda_2, \ldots, \lambda_r)$ and $\bar{\mu} = (\mu_2, \ldots, \mu_s)$. Then $d^{e,0}_{\lambda\mu}(v) = d^{e,0}_{\bar{\lambda}\bar{\mu}}(v)$.

Theorem (Donkin, 1998)

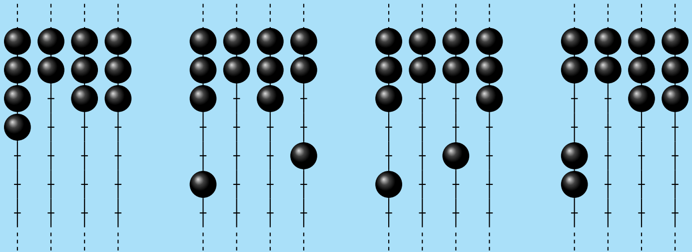
If λ, μ, $\bar{\lambda}$, and $\bar{\mu}$, are as in either case above, then $d^{e,p}_{\lambda\mu}(1) = d^{e,p}_{\bar{\lambda}\bar{\mu}}(1)$.

Abacus combinatorics

We may encode partition combinatorics in an abacus display. We demonstrate this by the following example.

Example

Let $e = 4 \rightsquigarrow$ 4-runner abacus. Let $\rho = (1^3)$ be a core, and $\kappa = (9^2, 1)$, $\lambda = (9, 8, 2)$, and $\mu = (9, 6, 2^2)$ be three partitions in the weight 4 block with core ρ.



Runner removal

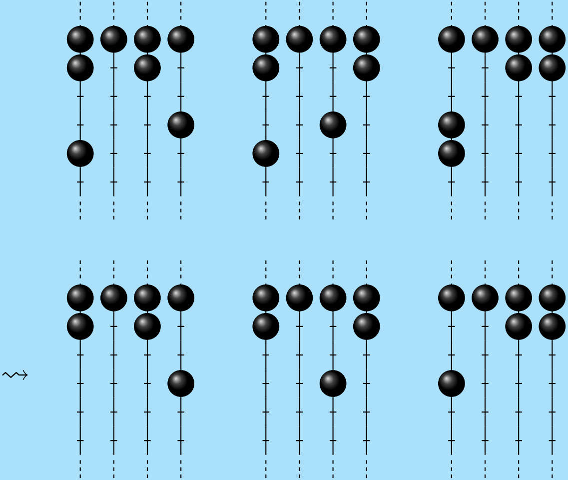
Theorem (James–Mathas, 2002)

Suppose $e \geqslant 3$, λ, μ: partitions of n, μ: e-regular, and take abacus displays for λ, μ. Suppose that the last bead on runner i (some i) occurs before every unoccupied space on both abacus displays $\rightsquigarrow$ define two abacus displays with $e-1$ runners by deleting runner i from those of λ, μ $\rightsquigarrow$ partitions λ^- and μ^-. If μ^- is $(e-1)$-regular, then

$$d^{e,0}_{\lambda\mu}(v) = d^{e-1,0}_{\lambda^-\mu^-}(v).$$

Example

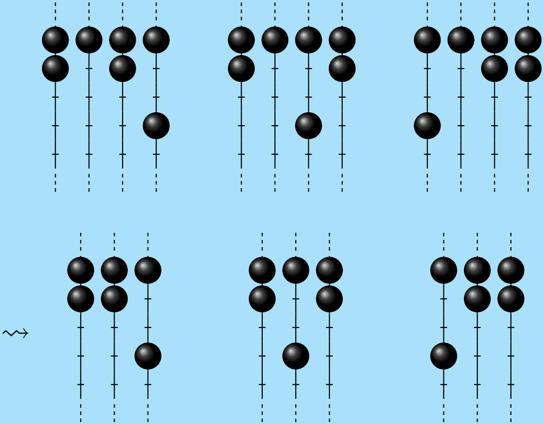
Let $e = 4$, $\kappa = (9^2, 1)$, $\lambda = (9, 8, 2)$, and $\mu = (9, 6, 2^2)$, as before. We'll first remove the first row from each partition.



We obtain $\bar{\kappa} = (9, 1)$, $\bar{\lambda} = (8, 2)$, and $\bar{\mu} = (6, 2^2)$.

Example (continued)

From $\bar{\kappa} = (9, 1)$, $\bar{\lambda} = (8, 2)$, and $\bar{\mu} = (6, 2^2)$, apply runner removal.



We obtain $\bar{\kappa}^- = (6)$, $\bar{\lambda}^- = (5, 1)$, and $\bar{\mu}^- = (4, 1^2)$. Then e.g.

$$d^{e,p}_{\lambda\kappa}(v) = d^{e,p}_{\bar{\lambda}\bar{\kappa}}(v) \text{ and } d^{e,0}_{\bar{\lambda}\bar{\kappa}}(v) = d^{e-1,0}_{(5,1)(6)}(v) = v.$$

Example (continued)

$\bar{\kappa} = (9,1)$, $\bar{\lambda} = (8,2)$, and $\bar{\mu} = (6,2^2)$.
$\bar{\kappa}^- = (6)$, $\bar{\lambda}^- = (5,1)$, and $\bar{\mu}^- = (4,1^2)$.

$$d^{e,p}_{\lambda\kappa}(v) = d^{e,p}_{\bar{\lambda}\bar{\kappa}}(v) \text{ and } d^{e,0}_{\bar{\lambda}\bar{\kappa}}(v) = d^{e-1,0}_{(5,1)(6)}(v) = v.$$

But we've only looked in characteristic 0! It is known that in weight 2 blocks, $d^{e,p}_{\lambda\mu}(v) = d^{e,0}_{\lambda\mu}(v)$ for any $p > 2$. But what about $p = 2$? Results of Richards & Fayers give the 'adjustment matrices' for such blocks, and we may check that we've chosen our partitions well enough so that we see the same submatrix, even if $p = 2$.

	6	$5,1$	$4,1^2$	3^2	$3,2,1$
6	1	·	·	·	·
$5,1$	v	1	·	·	·
$4,1^2$	·	v	1	·	·
3^2	·	v	·	1	·
$3,2,1$	v	v^2	v	v	1

Summary

First, we solve the problem for weight 2 and 3 blocks: We choose five partitions so that all the ‘action’ happens on the ‘longest three runners’. Then we can remove all but these three runners, using the runner removal result. We use the known adjustment matrices in weights 2 and 3 to choose our partitions well.

In a given weight $w \geqslant 4$ block, put $w - 2$ of the e-rim hooks in the first row, and choose the other 2 based on what should work for the ‘remaining’ weight 2 block. Row removal then gives the decomposition numbers we need.

This strategy *usually* works, and we may reduce to weight 2, so long as we don’t land in some degenerate cases, and then we have to argue in a different manner.

The remaining weight 2 case

What was wrong with the weight 2 Rouquier block when $e = 3$?

There are only 5 simples! The Gabriel quiver for these just looks like $D_4^{(1)}$ if $p \neq 2$, or A_5 if $p = 2$. We get that the block is Schurian-infinite from the $D_4^{(1)}$, and gave a separate argument to show that the block is Schurian-infinite if $p = 2$.

I don't know if it is strictly wild, or how to go about proving it!