2-roots for simply laced Weyl groups

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Outline

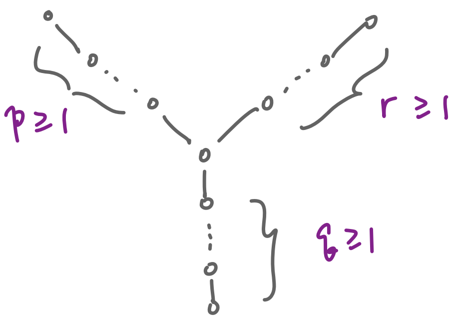
- **Background:** Weyl groups, reflection representations, and the module M.
- **2-roots:** canonical basis, actions, and orbits.
- **Selected results:** semisimplicity, sign-coherence, and highest 2-roots.

Convention / Assumptions:

- To simplify statements, we assume that $\Bbbk$ is a field w/ $\operatorname{char} \Bbbk = 0$.
- All Coxeter systems will be simply-laced.

The groups $Y_{p,q,r}$

We will be interested in the Coxeter groups of type "$Y_{p,q,r}$":



$\rightsquigarrow$ $W = W(Y_{p,q,r}) = \langle S \mid R \rangle$

S = the vertex set

$$R = \left\{ \begin{array}{lll} s^2 = 1 & \forall s \in S, & \\ st = ts & \text{if} & s \neq t, \\ sts = tst & \text{if} & s - t. \end{array} \right\}$$

Special cases:

- $Y_{1,1,r} = D_{r+3}$
- E_r, $\tilde{E}_r$ $(6 \leq r \leq 8)$, "E_n" $(n > 8)$.
- $Y_{344} / \langle \text{one extra rel} \rangle = C_2 \times \mathbb{M}$ (Monster)

The reflection representation

- Let $V = \bigoplus_{s \in S} k \underbrace{\alpha_s}_{\text{"simple roots"}}$ and let B be the bilinear form on V given by

$$B(\alpha_s, \alpha_t) = \begin{cases} 2 & \text{if } s = t, \\ -1 & \text{if } s - t, \\ 0 & \text{if } s \not- t. \end{cases}$$

- Facts: (a) V affords a reflection representation of W, given by

$$s \cdot \alpha_t = \alpha_t - \frac{2B(\alpha_t, \alpha_s)}{B(\alpha_s, \alpha_s)} \alpha_s = \begin{cases} -\alpha_t & \text{if } s = t, \\ \alpha_t + \alpha_s & \text{if } s - t, \\ \alpha_t & \text{if } s \not- t. \end{cases}$$

(b) invariance: $\forall g \in W,\ v, w \in V,\quad B(g \cdot v, g \cdot w) = B(v, w).$

The module M

- View B as a linear map $B : V \otimes V \to k$.

- Think of $S^2(V)$ as the subspace of $V \otimes V$ spanned by the elts

$$x \vee y := x \otimes y + y \otimes x .$$

- Now consider the restriction $B : S^2(V) \longrightarrow k$, and define

$$M := \ker B = \{ v \in S^2(V) : B(v) = 0 \}.$$

- Then M is a W-submodule of $S^2(V)$ with codimension 1.

Q: Is M simple? Completely reducible?

The canonical basis

Def: We define a 2-root (of W, V, etc) to be an element of the form $\alpha \vee \beta$ where α, β are roots in V w/ $B(\alpha, \beta) = 0$.

$w \cdot \alpha_s$, $w \in W$, $s \in S$.

E.g. Subgraph $\leadsto$ 2-root

a) nonadjacent $\overset{\circ}{i}$ $\overset{\circ}{j}$ — $\alpha_i \vee \alpha_j$

b) $i - j - k$ — $\alpha_j \vee (\alpha_i + \alpha_j + \alpha_k)$

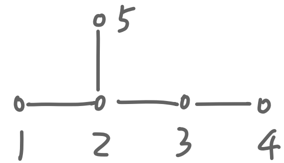
c) $1 - 2 - \cdots - m <^{s}_{t}$ — $\alpha_1 \vee \theta_1$, w/ $\theta_1 = \alpha_s + \alpha_m + \cdots + \alpha_2 + \alpha_1 + \alpha_2 + \cdots + \alpha_m + \alpha_t$

Prop. (Green–X.) The above 2-roots form a basis of M.

"the canonical basis, $\mathcal{B}$"

Example actions

D_5 :



$$s_2 \cdot (\alpha_1 \vee \alpha_3) = (s_2 \cdot \alpha_1) \vee (s_2 \cdot \alpha_3) = (\alpha_1 + \alpha_2) \vee (\alpha_3 + \alpha_2)$$

$$= \alpha_1 \vee \alpha_3 + \alpha_1 \vee \alpha_2 + \alpha_2 \vee \alpha_3 + \alpha_2 \vee \alpha_2$$

$$= \underbrace{\alpha_1 \vee \alpha_3}_{\beta} + \alpha_2 \vee \underbrace{(\alpha_1 + \alpha_2 + \alpha_3)}_{\beta}$$

$$s_3 \cdot \left(\alpha_3 \vee (\underbrace{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_2 + \alpha_5}_{\theta_3})\right) = (-\alpha_3) \vee (\alpha_1 + \alpha_2 + \alpha_3 - \alpha_3 + \alpha_2 + \alpha_3 + \alpha_5)$$

$$= \underline{-\alpha_3 \vee \theta_3}$$

More generally ...

Thm 1.(GX) (1) For any $\gamma \in S$ and $\alpha \vee \beta \in \mathcal{B}$, we have

$$S_\gamma \cdot (\alpha \vee \beta) = \begin{cases} \alpha \vee \beta & \text{if } B(\gamma, \alpha) = B(\gamma, \beta) = 0, \\ -\alpha \vee \beta & \text{if } \gamma \in \{\alpha, \beta\}, \\ \alpha \vee \beta + \gamma \vee v & \text{otherwise, for some } (\gamma \vee v) \in \mathcal{B}. \end{cases}$$

Moreover, in the last case we have $\gamma \vee v = g \cdot (\alpha \vee \beta)$ for some $g \in W$.

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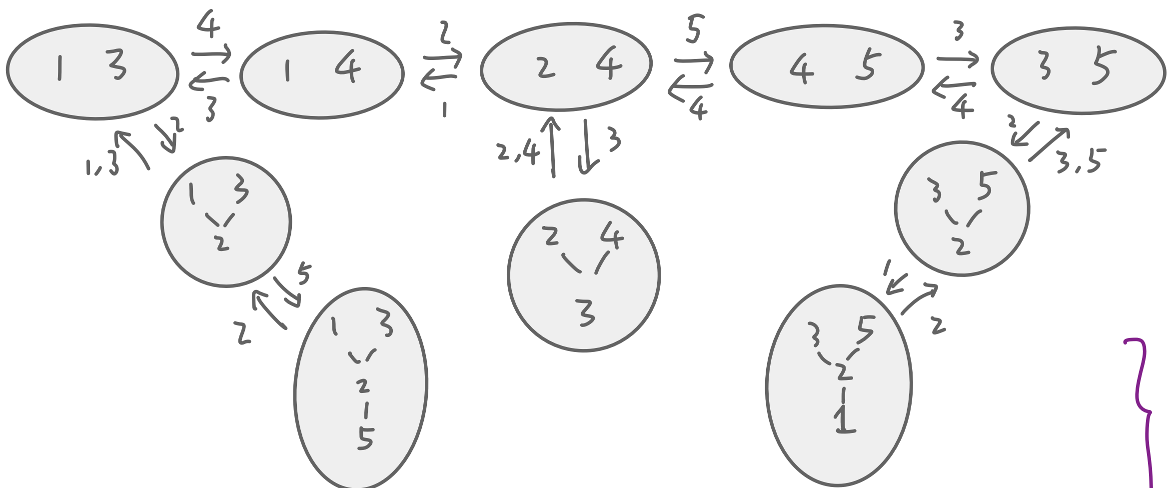
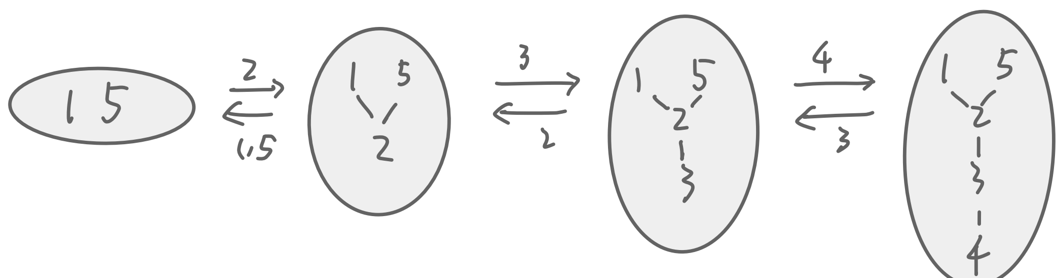
(2) We understand the orbits of the W-action on 2-roots, including the number of them and how they intersect with $\mathcal{B}$.

Example

$W = W(D_5) \rightsquigarrow |\mathcal{B}| = 14, \quad \#\text{ 2-roots} = 140, \quad \#\text{ orbits} = 2$





actions on $\mathcal{B}$ of the form

$$s_i \cdot x = x + y$$

$$(\cdots x \xrightarrow{i} y \cdots)$$

One motivation for studying $\mathcal{B}$: In type $Y_{1,q,r}$,

- M and $\mathcal{B}$ are specializations of certain Kazhdan-Lusztig (KL) cell modules and their KL bases, respectively.

Example: For any $W = W(E_n)$ with $n \geq 6$ and any left cell Γ of W with a-value 2, there is a natural bijection $\Gamma \to \mathcal{B}$, $w \mapsto b_w$

s.t. $\forall s \in S$, $w \in \Gamma$:

$$C_s \cdot [C_w] = \sum_{y \in \Gamma} \lambda_y \cdot [C_y]$$

$$\Updownarrow \qquad \Big\downarrow \text{"set } v = -1\text{"}$$

$$(s-1) \cdot b_w = \sum_{y \in \Gamma} \lambda_y \cdot b_y .$$

Semisimplicity

Facts: · V is irreducible $\Longleftrightarrow$ B is nondegenerate $\Longleftrightarrow$ $Y_{p,q,r} \notin \{\tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$.

· When B is nondegenerate, we can identify V^* with V and define

the Virasoro element $\omega = \sum_{s \in S} \alpha_s^* \vee \alpha_s$.

Thm 2. (GX) Suppose (W, S) is not affine. Then $S^2(V)$ and M are both completely reducible, with $S^2(V) = M \oplus k\omega$ and $M = \bigoplus_i k(\beta \cap X_i)$, where the X_i's are the W-orbits of 2-roots and $k(\beta \cap X_i)$ is a simple W-module for each i.

Sign-coherence

Prop (GX): Every 2-root of W is W-conjugate to an element in $\mathcal{B}$.

Thm 3 (GX): The basis $\mathcal{B}$ of M is column-sign-coherent:

$$\forall w \in W,\ b \in \mathcal{B},\quad w \cdot b \in \mathbb{Z}_{\geq 0}\mathcal{B} \cup \mathbb{Z}_{\leq 0}\mathcal{B}.$$

$\Updownarrow$

Thm 3' (GX): Every 2-root of W is a linear combination of elements of $\mathcal{B}$ with integer coefficients of like sign.

Highest 2-roots

Def: For $x, y \in \mathbb{Z}B$, define $x \le_2 y$ if $y - x \in \mathbb{Z}_{\ge 0}B$.

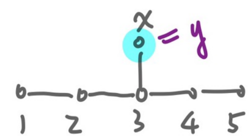
Thm 4. (GX) If W is finite, then every W-orbit of 2-roots contains a unique max. element w.r.t. $\le_2$.

Examples:

θ : h.r.

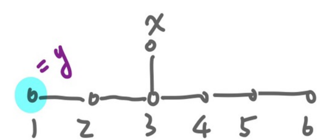
$\alpha_{ij} = \sum_{k \in p_{ij}} \alpha_k$

E_6



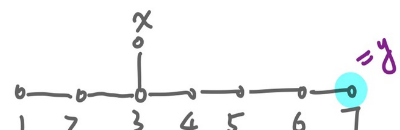
$(\theta - \alpha_{2,y}) \vee (\theta - \alpha_{4,y})$

E_7



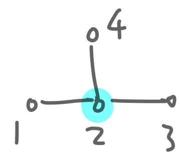
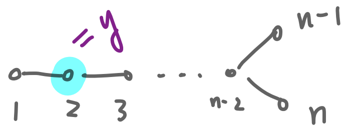
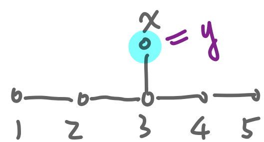
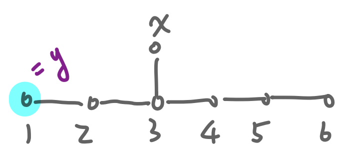
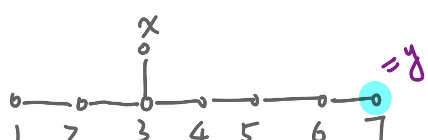
$(\theta - \alpha_{x,y}) \vee (\theta - \alpha_{4,y})$

E_8



$(\theta - \alpha_{2,y}) \vee (\theta - \alpha_{x,y})$

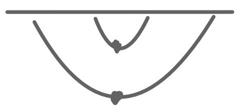
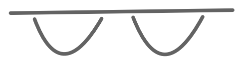
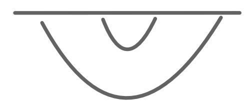
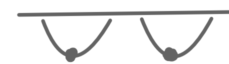
Thank you !

Type	Coxeter diagram	Orbit	Highest 2-root
D_4		"1" "3" "4"	$\alpha_{1,3} \vee \alpha_{1,4}$ $\alpha_{3,1} \vee \alpha_{3,4}$ $\alpha_{4,1} \vee \alpha_{4,3}$
D_n, $n>4$		small large	$\alpha_{1,n-1} \vee \alpha_{1,n}$ $(\theta - \alpha_{1,2}) \vee (\theta - \alpha_{2,3})$
E_6		—	$(\theta - \alpha_{2,y}) \vee (\theta - \alpha_{4,y})$
E_7		—	$(\theta - \alpha_{x,y}) \vee (\theta - \alpha_{4,y})$
E_8		—	$(\theta - \alpha_{2,y}) \vee (\theta - \alpha_{x,y})$

$\left(\alpha_{i,j} = \sum_{p \in P_{ij}} \alpha_p\right)$

Connection to Temperley-Lieb algebras

In type D_n (and A_n), the expansion of a 2-root into $\mathbb{B}$ dictates reductions in suitable Temperley-Lieb diagrams:

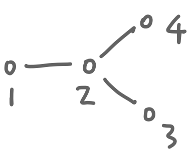
D_4 :  =  +  + 

$(\varepsilon_1+\varepsilon_4)\vee(\varepsilon_2+\varepsilon_3)$ — a 2-root

$(\varepsilon_1-\varepsilon_2)\vee(\varepsilon_3-\varepsilon_4) = \alpha_1\vee\alpha_3$

$(\varepsilon_2-\varepsilon_3)\vee(\varepsilon_1-\varepsilon_4) = \alpha_2\vee(\alpha_1+\alpha_2+\alpha_3)$

$(\varepsilon_3+\varepsilon_4)\vee(\varepsilon_1+\varepsilon_2) = \alpha_4\vee\theta_4$

(D_4 : 

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