Representation type of cyclotomic quiver Hecke algebras in affine type A[1]

Qi WANG
Yau Mathematical Sciences Center
Tsinghua University

Representation Theory of Hecke Algebras and Categorification

@OIST, June 5, 2023.

[1]This is joint work with Susumu Ariki and Linliang Song.

Outline

Introduction

Hecke algebras of type A

The symmetric group $\mathfrak{S}_n$ (= permutation group of $\{1, 2, \cdots, n\}$) is generated by $\{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$ subject to

$$s_i^2 = 1, (\Leftrightarrow (s_i + 1)(s_i - 1) = 0)$$

$$s_i s_j = s_j s_i \text{ if } |i - j| \neq 1, \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i - j| = 1.$$

Hecke algebras of type A

The symmetric group $\mathfrak{S}_n$ (= permutation group of $\{1, 2, \cdots, n\}$) is generated by $\{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$ subject to

$$s_i^2 = 1, (\Leftrightarrow (s_i + 1)(s_i - 1) = 0)$$

$$s_i s_j = s_j s_i \text{ if } |i - j| \neq 1, \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i - j| = 1.$$

The Iwahori-Hecke algebra $\mathcal{H}(\mathfrak{S}_n)$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\{T_i \mid 1 \leq i \leq n-1\}$ subject to

$$T_i^2 = (q-1)T_i + q, (\Leftrightarrow (T_i + 1)(T_i - q) = 0)$$

$$T_i T_j = T_j T_i \text{ if } |i - j| \neq 1, \quad T_i T_j T_i = T_j T_i T_j \text{ if } |i - j| = 1.$$

From the perspective of Lie theory, one wants to know

- the irreducible representations of $\mathcal{H}(\mathfrak{S}_n)$;
- the decomposition numbers of $\mathcal{H}(\mathfrak{S}_n)$.

This is accompanied by the rise of many theories, such as categorification theory, cellular algebra theory, crystal bases theory, Kazhdan-Lusztig theory, Lascoux-Leclerc-Thibon algorithm, etc.

From the perspective of Lie theory, one wants to know

- the irreducible representations of $\mathcal{H}(\mathfrak{S}_n)$;
- the decomposition numbers of $\mathcal{H}(\mathfrak{S}_n)$.

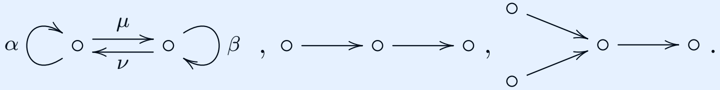
This is accompanied by the rise of many theories, such as categorification theory, cellular algebra theory, crystal bases theory, Kazhdan-Lusztig theory, Lascoux-Leclerc-Thibon algorithm, etc.

Now, we have many generalizations of $\mathcal{H}(\mathfrak{S}_n)$,

- Hecke algebras of Coxeter groups, i.e., of type B, D, E, etc.
- Cyclotomic Hecke algebras (a.k.a. Ariki-Koike algebras). See [Ariki-Koike, 1994], [Broue-Malle, 1993], and [Cherednik 1987].
- Cyclotomic quiver Hecke algebras (a.k.a. Cyclotomic KLR algebras). See [Khovanov-Lauda, 2009] and [Rouquier, 2008].

Quiver Representation Theory

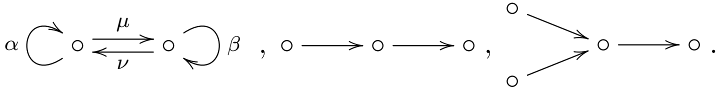
Quivers:



- paths: e.g., $(\alpha\mu\beta\nu)^m, (\mu\nu)^n\alpha^k, (\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Quiver Representation Theory

Quivers:



- paths: e.g., $(\alpha\mu\beta\nu)^m, (\mu\nu)^n\alpha^k, (\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Bound quiver algebras $A = KQ/I$:

$$I = \langle \sum \lambda_i \omega_i, \cdots \rangle$$

- $\lambda_i \in K$ and ω_i is a path but not an arrow.

Any (basic, connected) algebra A over K is isomorphic to a bound quiver algebra KQ/I.

Any (basic, connected) algebra A over K is isomorphic to a bound quiver algebra KQ/I.

From the perspective of algebraic representation theory, one wants to find all indecomposable rep's of KQ/I.

e.g., the quiver $\circ \longrightarrow \circ \longrightarrow \circ$ has 6 indecomposable rep's:

$$K \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad K \xrightarrow{1} K \xrightarrow{0} 0$$

$$0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} K \xrightarrow{1} K$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} K \qquad K \xrightarrow{1} K \xrightarrow{1} K$$

Representation type of algebras

Theorem (Drozd 1977)

The representation type of any algebra (over K) is exactly one of rep-finite, tame and wild.

Representation type of algebras

Theorem (Drozd 1977)

The representation type of any algebra (over K) is exactly one of rep-finite, tame and wild.

An algebra A is said to be

- **rep-finite** if the number of indecomposable rep's is finite.
- **tame** if it is not rep-finite, but all indecomposable rep's can be organized in a one-parameter family in each dimension.

Otherwise, A is called **wild**.

Some examples related to Hecke algebras.

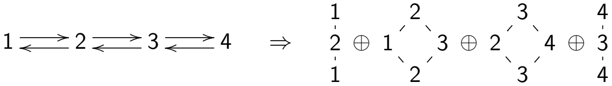
- rep-finite: e.g., Brauer tree algebras

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \quad \Rightarrow \quad \begin{matrix}1\\2\\1\end{matrix} \oplus \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \oplus \begin{matrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{matrix} \oplus \begin{matrix}4\\3\\4\end{matrix}$$

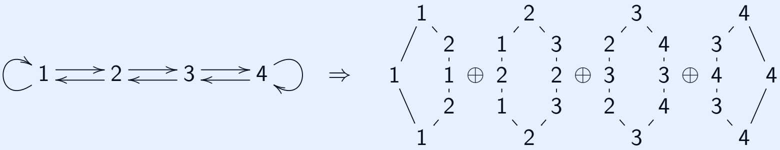
- tame: e.g., Brauer graph algebras
- wild:

Some examples related to Hecke algebras.

- rep-finite: e.g., Brauer tree algebras



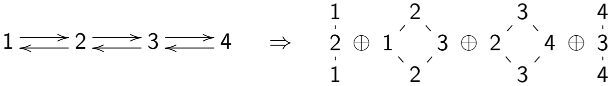
- tame: e.g., Brauer graph algebras



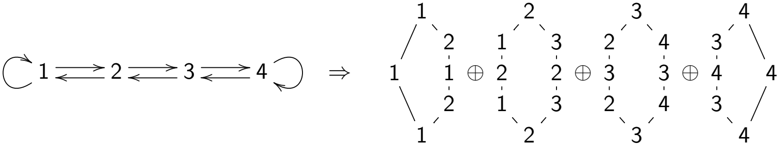
- wild:

Some examples related to Hecke algebras.

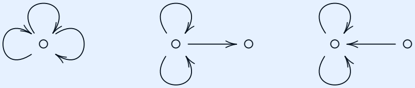
- rep-finite: e.g., Brauer tree algebras



- tame: e.g., Brauer graph algebras



- wild:



KLR algebras in affine type A

Cartan matrix of type $A_\ell^{(1)}$

Let $I = \{0, 1, ..., \ell\}$ be an index set.

- If $\ell = 1$, then
$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix};$$
- If $\ell \geq 2$, then
$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be the Cartan datum in type $A_\ell^{(1)}$, where

- $P = \bigoplus_{i=0}^{\ell} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ is the weight lattice;
- $\Pi = \{\alpha_i \mid i \in I\} \subset P$ is the set of simple roots;
- $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is the coweight lattice;
- $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ is the set of simple coroots.

The null root is $\delta = \alpha_0 + \alpha_1 + \ldots + \alpha_\ell$. We have

$$\langle h_i, \alpha_j \rangle = a_{ij}, \quad \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for all } i, j \in I.$$

We set $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$.

Quiver Hecke algebras

The quiver Hecke algebra $R(n)$ associated with $(Q_{i,j}(u,v))_{i,j\in I}$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \{x_i \mid 1 \leq i \leq n\}, \{\psi_j \mid 1 \leq j \leq n-1\},$$

subject to the following relations:

(1) $e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu),\ \sum_{\nu\in I^n} e(\nu) = 1,\ x_ix_j = x_jx_i,\ x_ie(\nu) = e(\nu)x_i.$

(2) $\psi_i e(\nu) = e(s_i(\nu))\psi_i,\ \psi_i\psi_j = \psi_j\psi_i$ if $|i-j| > 1$.

(3) $\psi_i^2 e(\nu) = Q_{\nu_i,\nu_{i+1}}(x_i, x_{i+1})e(\nu)$.

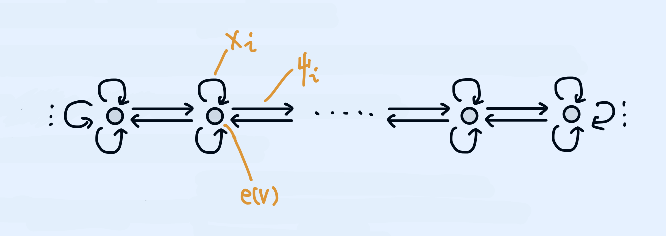
(4) $(\psi_i x_j - x_{s_i(j)}\psi_i)e(\nu) = \begin{cases} -e(\nu) & \text{if } j = i \text{ and } \nu_i = \nu_{i+1}, \\ e(\nu) & \text{if } j = i+1 \text{ and } \nu_i = \nu_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$

(5) $(\psi_{i+1}\psi_i\psi_{i+1} - \psi_i\psi_{i+1}\psi_i)e(\nu) = \begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i - x_{i+2}} e(\nu) & \text{if } \nu_i = \nu_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$

Quiver Hecke algebras

The quiver Hecke algebra $R(n)$ associated with $(Q_{i,j}(u,v))_{i,j\in I}$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \{x_i \mid 1 \le i \le n\}, \{\psi_j \mid 1 \le j \le n-1\},$$



Cyclotomic quiver Hecke algebras

Fix $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ w.r.t. Λ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

Cyclotomic quiver Hecke algebras

Fix $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ w.r.t. Λ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

Here, $R^\Lambda(n)$ is a finite-dimensional symmetric algebra proved by Shan-Varagnolo-Vasserot in 2017.

Cyclotomic quiver Hecke algebras

Fix $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ w.r.t. Λ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

Here, $R^\Lambda(n)$ is a finite-dimensional symmetric algebra proved by Shan-Varagnolo-Vasserot in 2017.

Let $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For each $\beta \in Q_+$ with $|\beta| = n$, we define

$$R^\Lambda(\beta) := e(\beta) R^\Lambda(n) e(\beta),$$

where $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ with $I^\beta = \left\{ \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n \mid \sum_{i=1}^{n} \alpha_{\nu_i} = \beta \right\}$.

An example

Set $\Lambda = k\Lambda_0, \ell = 2$. Then, $I = \{0, 1, 2\}$ and $R(3)$ is generated by

$$\{e(000), \cdots, e(012), \cdots, e(212), \cdots\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to the relations.

An example

Set $\Lambda = k\Lambda_0, \ell = 2$. Then, $I = \{0, 1, 2\}$ and $R(3)$ is generated by

$$\{e(000), \cdots, e(012), \cdots, e(212), \cdots\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to the relations.

Set $\beta = \alpha_0 + \alpha_1 + \alpha_2$. Then, $R^\Lambda(\beta)$ is generated by

$$\{e(012), e(021), e(102), e(120), e(201), e(210)\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to

- $e(102) = e(120) = e(201) = e(210) = 0$, $x_1^k e(012) = x_1^k e(021) = 0$;
- $\psi_1 e(012) = \psi_1 e(021) = 0$, $\psi_2 e(012) = e(021)\psi_2$;
- $x_2 e(012) = -x_1 e(012)$, $x_2 e(021) = -t x_1 e(021)$;
- $x_3^2 e(012) = t x_1^2 e(012) + (1-t) x_1 x_3 e(012)$, etc.

Known results on cyclotomic KLR algebras

We know the representation type of cyclotomic KLR algebras in the following cases.

- $R^{\Lambda_0}(\beta)$ in type $A^{(2)}_{2\ell}$, see [Ariki-Park, 2014].
- $R^{\Lambda_0}(\beta)$ in type $A^{(1)}_{\ell}$, see [Ariki-Iijima-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $C^{(1)}_{\ell}$, see [Ariki-Park, 2015].
- $R^{\Lambda_s}(\beta)$ in type $C^{(1)}_{\ell}$, see [Chung-Hudak, 2023].
- $R^{\Lambda_0}(\beta)$ in type $D^{(2)}_{\ell+1}$, see [Ariki-Park, 2016].
- $R^{\Lambda_0+\Lambda_s}(\beta)$ in type $A^{(1)}_{\ell}$, see [Ariki, 2017].

In this talk, we explain the representation type of $R^{\Lambda}(\beta)$ in type $A^{(1)}_{\ell}$, for arbitrary $\Lambda \in P^+$.

Representation type of $R^\Lambda(\beta)$

- Rickard's result tells us that two symmetric algebras have the same representation type if they are derived equivalent.

Representation type of $R^\Lambda(\beta)$

- Rickard's result tells us that two symmetric algebras have the same representation type if they are derived equivalent.
- Chuang and Rouquier's result tells us that $R^\Lambda(\beta)$ and $R^\Lambda(\beta')$ are derived equivalent if $\Lambda - \beta$ and $\Lambda - \beta'$ lie in the same W-orbit of the set $P(\Lambda)$ of weights of $V(\Lambda)$, where W is the affine symmetric group generated by (for $i \in I$)

  $$s_i^2 = 1, s_i s_j = s_j s_i \text{ if } |i-j| \not\equiv_{\ell+1} 1, s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| \equiv_{\ell+1} 1.$$

Representation type of $R^\Lambda(\beta)$

- Rickard's result tells us that two symmetric algebras have the same representation type if they are derived equivalent.
- Chuang and Rouquier's result tells us that $R^\Lambda(\beta)$ and $R^\Lambda(\beta')$ are derived equivalent if $\Lambda - \beta$ and $\Lambda - \beta'$ lie in the same W-orbit of the set $P(\Lambda)$ of weights of $V(\Lambda)$, where W is the affine symmetric group generated by (for $i \in I$)

$$s_i^2 = 1, s_i s_j = s_j s_i \text{ if } |i-j| \not\equiv_{\ell+1} 1, s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| \equiv_{\ell+1} 1.$$

- Kac's result tells us that the representatives of W-orbits in $P(\Lambda)$ are given by $\{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}$, where

$$\max^+(\Lambda) := \{\mu \in P^+ \mid \mu \text{ is maximal}\}.$$

A weight $\mu \in P(\Lambda)$ is maximal if $\mu + \delta \notin P(\Lambda)$.

$\max^{+}(\Lambda)$

We briefly recall the construction in [Kim-Oh-Oh, 2020] as follows.

Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^{+}$. We define

$$\mathrm{le}(\Lambda) = \sum a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) = i_1 + i_2 + \cdots + i_n.$$

Suppose $\mathrm{le}(\Lambda) = k$. Then,

$$P^{+}_{cl,k}(\Lambda) = \left\{ \Lambda' \in P^{+} \mid \mathrm{le}(\Lambda) = \mathrm{le}(\Lambda'), \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

$\max^+(\Lambda)$

We briefly recall the construction in [Kim-Oh-Oh, 2020] as follows.

Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^+$. We define

$$\mathrm{le}(\Lambda) = \sum a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) = i_1 + i_2 + \cdots + i_n.$$

Suppose $\mathrm{le}(\Lambda) = k$. Then,

$$P^+_{cl,k}(\Lambda) = \left\{\Lambda' \in P^+ \mid \mathrm{le}(\Lambda) = \mathrm{le}(\Lambda'), \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

e.g., $P^+_{cl,3}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, $2\Lambda_0 + \Lambda_2$, $\Lambda_4 + 2\Lambda_6$, $2\Lambda_5 + \Lambda_6$, $\Lambda_0 + 2\Lambda_1$, $2\Lambda_2 + \Lambda_5$, $\Lambda_1 + 2\Lambda_4$, $2\Lambda_0 + \Lambda_2$, $3\Lambda_3$.

Theorem (Kim-Oh-Oh 2020)

For any $\Lambda \in P^+_{cl,k}$, there is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P^+_{cl,k}(\Lambda)$.

Theorem (Kim-Oh-Oh 2020)

For any $\Lambda \in P^+_{cl,k}$, there is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P^+_{cl,k}(\Lambda)$.

Remaining: to obtain the inverse ϕ_Λ^{-1}.

Theorem (Kim-Oh-Oh 2020)

For any $\Lambda \in P^+_{cl,k}$, there is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P^+_{cl,k}(\Lambda)$.

Remaining: to obtain the inverse ϕ_Λ^{-1}.

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

Theorem (Kim-Oh-Oh 2020)

For any $\Lambda \in P^+_{cl,k}$, there is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P^+_{cl,k}(\Lambda)$.

Remaining: to obtain the inverse ϕ_Λ^{-1}.

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

Then, we consider the linear equation $AX^t = Y^t_{\Lambda'}$.

Proposition (Ariki-Song-W. 2023)

- The linear equation $AX^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, \ldots, x_\ell)$ satisfying

$$X \in \mathbb{Z}_{\geq 0}^{\ell+1} \text{ and } \min\{x_i\} = 0.$$

Proposition (Ariki-Song-W. 2023)

- The linear equation $AX^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, \dots, x_\ell)$ satisfying

$$X \in \mathbb{Z}_{\geq 0}^{\ell+1} \text{ and } \min\{x_i\} = 0.$$

- The inverse map $\phi_\Lambda^{-1} : P^+_{cl,k}(\Lambda) \to \max^+(\Lambda)$ of ϕ_Λ is given by

$$\phi_\Lambda^{-1}(\Lambda') = \Lambda - \sum_{i \in I} x_i \alpha_i,$$

where X is the unique solution of $AX^t = Y^t_{\Lambda'}$ as above.

Proposition (Ariki-Song-W. 2023)

- The linear equation $AX^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, \ldots, x_\ell)$ satisfying

$$X \in \mathbb{Z}^{\ell+1}_{\geq 0} \text{ and } \min\{x_i\} = 0.$$

- The inverse map $\phi_\Lambda^{-1} : P^+_{cl,k}(\Lambda) \to \max^+(\Lambda)$ of ϕ_Λ is given by

$$\phi_\Lambda^{-1}(\Lambda') = \Lambda - \textstyle\sum_{i\in I} x_i\alpha_i,$$

where X is the unique solution of $AX^t = Y^t_{\Lambda'}$ as above.

Set $\beta_{\Lambda'} := \sum_{i\in I} x_i\alpha_i$. Then,

$$\max^+(\Lambda) = \left\{\Lambda - \beta_{\Lambda'} \mid \Lambda' \in P^+_{cl,k}(\Lambda)\right\}.$$

Strategy to prove the results

If $\Lambda - \beta$ lies in the W-orbit of $P(\Lambda)$, then

$$\Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P^+_{cl,k}(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

Thus, we only need to consider $R^\Lambda(\beta)$ for $\beta = \beta_{\Lambda'} + m\delta$ with $\Lambda' \in P^+_{cl,k}(\Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$.

Strategy to prove the results

If $\Lambda - \beta$ lies in the W-orbit of $P(\Lambda)$, then

$$\Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P^+_{cl,k}(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

Thus, we only need to consider $R^\Lambda(\beta)$ for $\beta = \beta_{\Lambda'} + m\delta$ with $\Lambda' \in P^+_{cl,k}(\Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$.

Step 1: We show that $R^\Lambda(\beta_{\Lambda'} + m\delta)$ is wild for all $m \geq 1$ if $\beta_{\Lambda'} \neq 0$ and $R^\Lambda(m\delta)$ is wild for all $m \geq 2$, by using some new reduction theorems.

(If $R^\Lambda(\gamma)$ is not wild, we set $\gamma \in \mathcal{NW}(\Lambda) \cup \{\delta\}$.)

Step 2: We determine the representation type of $R^\Lambda(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^\Lambda(\gamma)$.)

Step 2: We determine the representation type of $R^\Lambda(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^\Lambda(\gamma)$.)

Step 3: We show that

$$\mathcal{NW}(\Lambda) \subset \mathcal{T}(\Lambda)$$

via case-by-case consideration on small k (i.e., $k = 3, 4, 5, 6$) and via induction on $k \geq 7$.

Structure of $P^{+}_{cl,k}(\Lambda)$

(in type $A^{(1)}_{\ell}$)

Recall that

$$\max^+(\Lambda) = \left\{\Lambda - \beta_{\Lambda'} \mid \Lambda' \in P^+_{cl,k}(\Lambda)\right\}.$$

e.g., $P^+_{cl,3}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, etc.

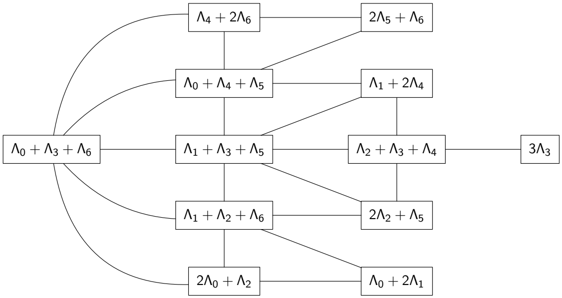
For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $k \geq 2$, we can write $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda}$ for some $i, j \in I$ and $\tilde{\Lambda} \in P^+_{cl,k-2}$. Then, we define

$$\Lambda'_{i,j} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda}.$$

Note that $\Lambda'_{i,j} = \Lambda'$ if and only if $j \equiv_e i - 1$.

Definition 3.1
Let $C(\Lambda)$ be an undirected graph, where we draw an edge between Λ' and Λ'' if $\Lambda'' = \Lambda'_{i,j}$ for some $i, j \in I$ with $j \not\equiv_e i - 1$.

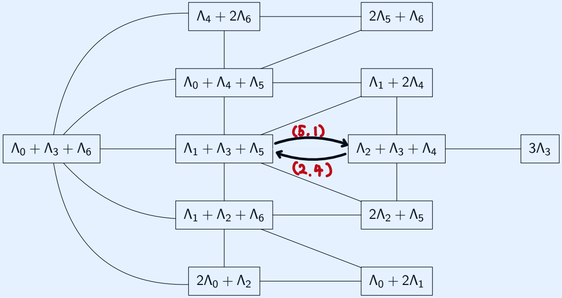
e.g., $C(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as follows.



Definition 3.1

Let $C(\Lambda)$ be an undirected graph, where we draw an edge between Λ' and Λ'' if $\Lambda'' = \Lambda'_{i,j}$ for some $i, j \in I$ with $j \not\equiv_e i - 1$.

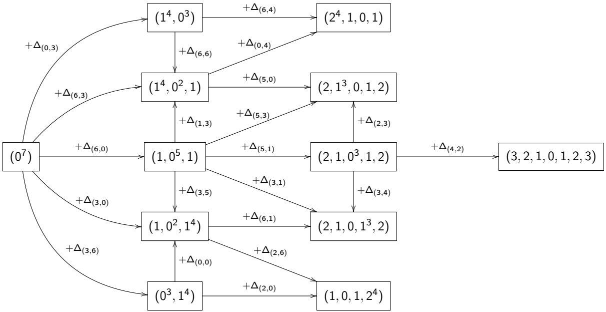
e.g., $C(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as follows.



We define

$$\Delta_{i,j} = \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \le j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

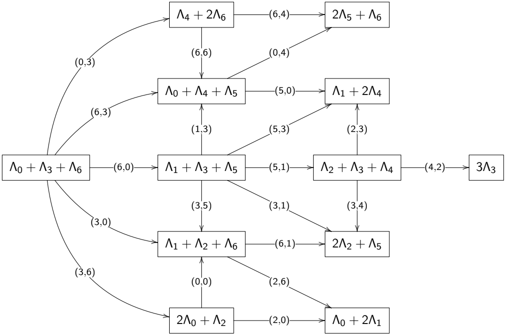
The unique solution of $AX^t = Y^t_{\Lambda''}$ is given by $\min(X_{\Lambda'} + \Delta_{i,j}) = 0$.
e.g.,



Definition 3.2

Let $\vec{C}(\Lambda)$ be the quiver where we set $\Lambda' \to \Lambda''$ if $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i,j}$. We label this arrow by (i,j).

e.g., $\vec{C}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as



Proposition 3.3

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $\Lambda' \neq \Lambda$, there is a directed path from Λ to Λ' in $\vec{C}(\Lambda)$. In particular, $\vec{C}(\Lambda)$ is a finite-connected quiver.

Proposition 3.3

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $\Lambda' \neq \Lambda$, there is a directed path from Λ to Λ' in $\vec{C}(\Lambda)$. In particular, $\vec{C}(\Lambda)$ is a finite-connected quiver.

Proposition 3.4

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. Then, there is a directed path

$$\Lambda^{(1)} \xrightarrow{(i_1,j_1)} \Lambda^{(2)} \xrightarrow{(i_2,j_2)} \dots \xrightarrow{(i_{m-1},j_{m-1})} \Lambda^{(m)} \ \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \xrightarrow{(i_1,j_1)} \Lambda^{(2)} + \tilde{\Lambda} \xrightarrow{(i_2,j_2)} \dots \xrightarrow{(i_{m-1},j_{m-1})} \Lambda^{(m)} + \tilde{\Lambda} \ \in \vec{C}(\Lambda).$$

Key Lemmas

Lemma 3.5

Suppose that there is an arrow $\Lambda' \xrightarrow{(i,j)} \Lambda''$ in $\vec{C}(\Lambda)$. If $R^\Lambda(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^\Lambda(\beta_{\Lambda''})$.

Key Lemmas

Lemma 3.5

Suppose that there is an arrow $\Lambda' \xrightarrow{(i,j)} \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

Lemma 3.6

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

Rep-finite and tame sets

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

Rep-finite and tame sets

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

For any $1 \leq j \leq h$, we define

$$F(\Lambda)_0 := \{\Lambda_{i_j,i_j} \mid m_{i_j} = 2\}$$
$$F(\Lambda)_1 := \{\Lambda_{i_j,i_{j+1}} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\}$$
$$T(\Lambda)_1 := \{\Lambda_{i_j,i_{j+1}} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\}$$
$$T(\Lambda)_2 := \{(\Lambda_{i_j,i_j})_{i_j-1,i_j+1} \mid m_{i_j} = 2, i_{j-1} \not\equiv_e i_j - 1, i_{j+1} \not\equiv_e i_j + 1\} \text{ if } \operatorname{char} K \neq 2$$
$$T(\Lambda)_3 := \{(\Lambda_{i_j,i_j})_{i_j,i_j+1 \text{ or } i_j-1,i_j} \mid m_{i_j} = 3, i_{j+1} \not\equiv_e i_j + 1 \text{ or } i_{j-1} \not\equiv_e i_j - 1\} \text{ if } \operatorname{char} K \neq 3$$
$$T(\Lambda)_4 := \{(\Lambda_{i_j,i_j})_{i_j,i_j} \mid m_{i_j} = 4\} \text{ if } \operatorname{char} K \neq 2$$
$$T(\Lambda)_5 := \{(\Lambda_{i_j,i_j})_{i_p,i_p} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_e i_j \pm 1, j \neq p\}$$

Set

$$\mathcal{F}(\Lambda) = \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$

$$\mathcal{T}(\Lambda) = \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1\leq j\leq 5} T(\Lambda)_j\}.$$

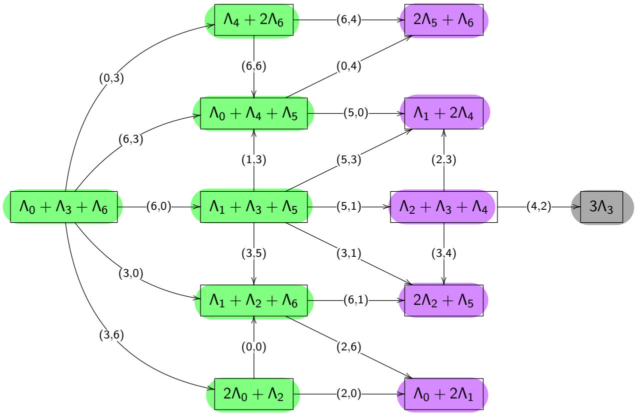
Theorem 3.7 (Ariki-Song-W. 2023)

Suppose $\mathrm{le}(\Lambda) \geq 3$. Then, $R^\Lambda(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

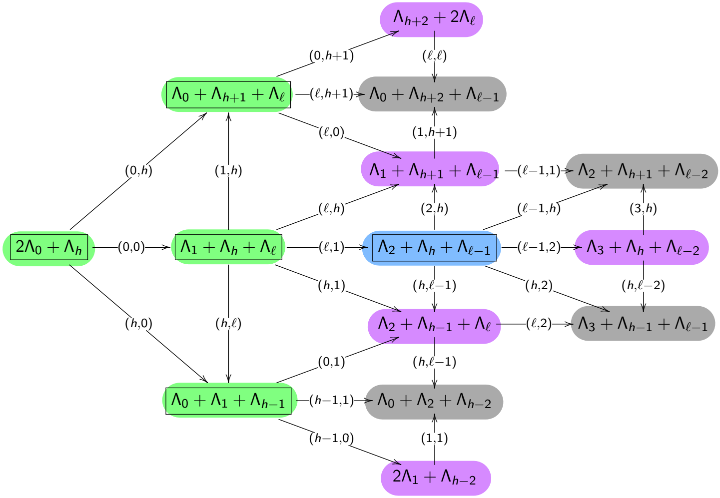
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0+\Lambda_3+\Lambda_6)$ with $\ell = 6$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_0 + \Lambda_h)$ is displayed as



References

[A17] S. Ariki, Representation type for block algebras of Hecke algebras of classical type. *Adv. Math.* **317** (2017), 823–845.

[AP16] S. Ariki and E. Park, Representation type of finite quiver Hecke algebras of type $D^{(2)}_{\ell+1}$. *Trans. Amer. Math. Soc.* **368** (2016), 3211–3242.

[KK12] S.-J. Kang and M. Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras. *Invent. Math.* **190 (3)** (2012), 699–742.

[KOO20] Young-Hun Kim, se-jin Oh and Young-Tak Oh, Cyclic sieving phenomenon on dominant maximal weights over affine Kac-Moody algebras. *Adv. Math.* **374** (2020), 107336.

[KL09] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups, I. *Represent. Theory* **13** (2009), 309–347.

[R08] R. Rouquier, 2-Kac-Moody algebras. Preprint (2008), arXiv: 0812.5023.

[S06] A. Skowroński, Selfinjective algebras: finite and tame type, Trends in Representation Theory of Algebras and Related Topics, 169–238, Contemp. Math. *Amer. Math. Soc.* **406**, 2006.

Thank you! Any questions?

Tools $\begin{cases} \text{Symmetric groups and Hecke algebras;} \\ \text{Bound quiver algebras;} \\ \text{Representation type: rep-finite, tame, wild;} \\ \text{Brauer tree/graph algebras.} \end{cases}$

Objects $\begin{cases} \text{Lie theoretic data;} \\ \text{Quiver Hecke algebras;} \\ \text{Cyclotomic KLR algebras;} \\ \max^{+}(\Lambda) \text{ and } P^{+}_{cl,k}(\Lambda); \\ \text{Rep-finite and tame sets.} \end{cases}$