CRITICAL CONVOLUTION ALGEBRAS
and
QUANTUM LOOP GROUPS

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1) CONVOLUTION ALGEBRAS ASSOCIATED to a QUIVER

2) K-THEORETICAL CRITICAL CONVOLUTION ALGEBRAS

3) KCA's and QUANTUM LOOP GROUPS

4) KCA's and REPRESENTATIONS, I

5) KCA's and REPRESENTATIONS, II

1) CONVOLUTION ALGEBRAS ASSOCIATED TO A QUIVER

$Q = (I, Q_1)$ quiver $\rightsquigarrow$ two different algebras

- Nakajima convolution algebras $\leftrightarrow$ q-loop gps
- K-theoretical Hall algebras $\leftrightarrow$ half q-loop gps

From now on : Q of type ADE, $R = \mathbb{C}[q, q^{-1}]$, $F = \mathbb{C}(q)$

Nakajima convolution algebras

$Q \rightsquigarrow$ [N, 94, 98, 01]

- quiver variety, Steinberg analogue : $\mathfrak{M}(W)$, $Z(W)$
- $K^{G_W \times \mathbb{C}^\times}(Z(W))$ a convolution algebra
- $U_F(L\underset{\substack{\uparrow \\ Q}}{g}) \to K^{G_W \times \mathbb{C}^\times}(Z(W)) \otimes_R F$ algebra map

$$Q = \circ \longrightarrow \circ \qquad \overline{Q}_f = \begin{tikzcd} \square \arrow[d, shift left] & \square \arrow[d, shift left] \\ \circ \arrow[u, shift left] \arrow[r, shift left] & \circ \arrow[u, shift left] \arrow[l, shift left] \end{tikzcd}$$

= the double of the framed quiver

V, W = I-graded $\qquad \dim = v, w \in \mathbb{N}I$

$\overline{X}(V,W)$ = representation variety of $\overline{Q}_f$:

$$\overline{Q}_f = \begin{tikzcd} \square \arrow[d, shift left, "a_1^*"] & \square \arrow[d, shift left, "a_2^*"] \\ \circ \arrow[u, shift left, "a_1"] \arrow[r, shift left, "\alpha"] & \circ \arrow[u, shift left, "a_2"] \arrow[l, shift left, "\alpha^*"] \end{tikzcd} \rightsquigarrow x = \begin{tikzcd} W_1 \arrow[d, shift left, "x_{a_1^*}"] & W_2 \arrow[d, shift left, "x_{a_2^*}"] \\ V_1 \arrow[u, shift left, "x_{a_1}"] \arrow[r, shift left, "x_\alpha"] & V_2 \arrow[u, shift left, "x_{a_2}"] \arrow[l, shift left, "x_{\alpha^*}"] \end{tikzcd} \in \overline{X}(V,W)$$

$$G_V \times G_W \times \mathbb{C}^* \curvearrowright \overline{X}(V,W) \qquad G_V = \prod_{i \in I} GL(V_i) \qquad G_W = \prod_{i \in I} GL(W_i)$$

$\overline{X}(V,W)$ holomorphic symplectic variety, moment map:

$$\mu_v : \overline{X}(V,W) \longrightarrow \mathfrak{g}_v^\vee \qquad x \longmapsto [x_\alpha, x_{\alpha^*}] + x_{a^*} x_a$$

$$\mathfrak{M}(v,w) = \mu_v^{-1}(0)_s / G_V \qquad \mathfrak{M}_0(v,w) = \mu_v^{-1}(0) /\!/ G_V$$

smooth, quasi projective

$\pi : \mathfrak{M}(v,w) \longrightarrow \mathfrak{M}_0(v,w)$ projective map

$\mathfrak{M}(w) = \bigsqcup_{v \in \mathbb{N}I} \mathfrak{M}(v,w)$ $\qquad Z(w) = \mathfrak{M}(w) \times_{\mathfrak{M}_0(w)} \mathfrak{M}(w)$

monoidal structure on $D^b \mathrm{Coh}_{G_w \times \mathbb{C}^\times}(\mathfrak{M}^2(w))_{Z(w)}$

set theoretic support

$$\begin{tikzcd}
 & \mathfrak{M}(w) \times \mathfrak{M}(w) \times \mathfrak{M}(w) \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
\underset{\mathcal{E}}{\mathfrak{M}(w) \times \mathfrak{M}(w)} & \underset{\mathcal{E} * \mathcal{F}}{\mathfrak{M}(w) \times \mathfrak{M}(w)} & \underset{\mathcal{F}}{\mathfrak{M}(w) \times \mathfrak{M}(w)}
\end{tikzcd}$$

$$\mathcal{E} * \mathcal{F} = (R\pi_{13})_* \left((L\pi_{12})^* (\mathcal{E}) \overset{L}{\otimes} (L\pi_{23})^* (\mathcal{F}) \right)$$

$\rightsquigarrow$

$\boxed{\text{algebra structure on } K^{G_w \times \mathbb{C}^\times}(Z(w))}$

$+ \quad \boxed{\text{natural action on } K^{G_w \times \mathbb{C}^\times}(\mathfrak{M}(w)),\ K^{G_w \times \mathbb{C}^\times}(\mathcal{L}(w))}$

$+ \quad \boxed{U_F(Lg) \xrightarrow[\text{homomorphism}]{\text{algebra}} K^{G_w \times \mathbb{C}^\times}(Z(w)) \otimes_R F}$

Reminders on critical K-theory [Orlov, Efimov-Positselski, Hirano, ...]

LG G-invariant model
$$\begin{cases} X = \text{smooth quasi proj variety} / \mathbb{C}, \\ G \text{ affine algebraic group} \curvearrowright X \\ \phi : X \to \mathbb{C} \text{ } G\text{-invariant regular function} \\ \mathrm{crit}(\phi) \subseteq \phi^{-1}(0) \end{cases}$$

$Z \subset \phi^{-1}(0)$ closed

$$(X, \phi) \rightsquigarrow \mathrm{DCoh}_G(X, \phi)_Z := D^b \mathrm{Coh}_G(\phi^{-1}(0))_Z \Big/ \mathrm{Perf}_G(\phi^{-1}(0))_Z$$

<u>equivariant category of singularities of (X, ϕ)</u>

$$\boxed{K_G(X, \phi)_Z := K_0(\mathrm{DCoh}_G(X, \phi)_Z)} = \underline{\text{critical K-theory}}$$

- $\phi^{-1}(0)$ smooth $\Rightarrow$ $D^b(\mathrm{Coh}_G(\phi^{-1}(0))) = \mathrm{Perf}_G(\phi^{-1}(0))$

  $\rightsquigarrow$ $K_G(X, \phi)$ supported on $\mathrm{crit}(\phi)$

- $\phi = 0$ up to replace $\phi^{-1}(0)$ by the derived zero locus of ϕ

  $K_G(X, 0) = K_G(X)$

K-theoretical Hall algebras

$$\begin{cases} Q \text{ quiver} \\ p \text{ potential} = \text{finite linear combination of cyclic words of } Q \\ \text{grading on } Q \text{ s.t. } p \text{ homogeneous of degree } 0 \end{cases}$$

$\rightsquigarrow X$, $\mathbb{C}^\times$-action on X, $\phi := \mathrm{tr}(p) : X \to \mathbb{C}$

$$\boxed{(Q,p) \underset{[\text{Pădurariu, '22}]}{\rightsquigarrow} \text{K-theoretical Hall algebra}}$$

$v \in \mathbb{N}^I$ $\qquad \mathfrak{X}(v) = \left[X(v) / G_v \right]$ $\qquad \mathfrak{X} = \bigsqcup_{v \in \mathbb{N}^I} \mathfrak{X}(v)$

monoidal structure on $\mathrm{DCoh}_{G \times \mathbb{C}^*}(X, \phi)$

q smooth

p proper

$$\begin{tikzcd} & \mathfrak{X}^{(2)} = \{x \subset y\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{X} \times \mathfrak{X} & & \mathfrak{X} \\ \mathcal{E}, \mathcal{F} & & \mathcal{E} * \mathcal{F} \end{tikzcd}
\qquad
\begin{tikzcd} & x \subset y \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x, y/x & & y \end{tikzcd}$$

$$\mathcal{E} * \mathcal{F} = (Rp)_* (Lq)^* (\mathcal{E} \boxtimes \mathcal{F})$$

$$\boxed{K_{\mathbb{C}^\times}(\mathfrak{X}, \phi) = K_{G \times \mathbb{C}^*}(X, \phi)} = \underline{\text{K-theoretical Hall algebra}}$$

Let $\tilde{Q}$ = triple quiver associated to Q

$Q = \circ \longrightarrow \circ \qquad \tilde{Q} = \varepsilon \circlearrowleft \circ \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} \circ \circlearrowright \varepsilon$

Theorem (V-V) Let Q of Dynkin type (or affine). Consider $(\tilde{Q}, p)$ where $p = \varepsilon[\alpha, \alpha^*]$. Then $\mathcal{U}_R(Lg)^+ \simeq K_{\mathbb{C}^\times}(\tilde{\mathfrak{X}}, \phi)$

AIM

- (Q, p) or $(X, \phi) \rightsquigarrow$ $\boxed{\text{critical convolution algebra}}$

  generalizes Nakajima's one

- $(\tilde{Q}_f, \tilde{p})$ link with quantum loop gps

  KCA = "double" of KHA

  different "doubles" of the same KHA are possible

  shifted q loop gps enter in the game

- $\boxed{\text{Consequences on representations}}$ geometric realization of some simple modules for (shifted) q loop gps

2) K-THEORETICAL CRITICAL CONVOLUTION ALGEBRAS

X smooth, $G \curvearrowright X$, $\phi : X \to \mathbb{C}$ regular G-inv, $\mathrm{crit}(\phi) \subseteq \phi^{-1}(0)$

X_0 = affine variety

$\pi : X \longrightarrow X_0$ G-equivariant proper

$\phi_0 : X_0 \longrightarrow \mathbb{C}$ regular, G-inv $\quad \phi_0 \circ \pi = \phi \quad x_0 \in \phi_0^{-1}(0)^G$

Set :

$Z = X \times_{X_0} X \qquad L = X \times_{X_0} \{x_0\} \qquad \phi^{(2)} := \phi \oplus (-\phi) : X^2 \longrightarrow \mathbb{C}$

$\rightsquigarrow Z \subseteq (\phi^{(2)})^{-1}(0)$

There exists a monoidal structure on $DCoh_G(X^2, f^{(2)})_Z$

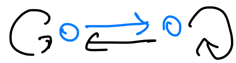
$\rightsquigarrow$ $\boxed{\begin{aligned} K_G(X^2, \phi^{(2)})_Z &= K_G(pt)\text{-algebra} \\ &= \underline{K\text{-th}^{al} \text{ critical convolution algebra}} \\ &\curvearrowright K_G(X,\phi),\ K_G(X,\phi)_L \end{aligned}}$

$+$

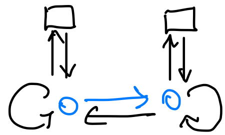
$\boxed{\begin{aligned} &\exists \text{ algebra homomorphism} \\ &K^G(Z) \longrightarrow K_G(X^2, \phi^{(2)})_Z \end{aligned}}$

3) KCA's and QUANTUM LOOP GROUPS

$\tilde{Q}$ = triple quiver of Q



$\tilde{Q}_f$ = framed triple quiver of Q



$$\tilde{\mathfrak{M}}(v,w) = \tilde{X}(V,W)_s / G_V$$

$$\tilde{\mathfrak{M}}_0(v,w) = \tilde{X}(V,W) /\!\!/ G_V$$

$$G_W \times \mathbb{C}^* \curvearrowright \tilde{\mathfrak{M}}(w), \tilde{\mathfrak{M}}_0(w)$$

grading on $\tilde{Q}_{f,1}$
$$\begin{cases} \alpha, \alpha^*, a, a^* & : -T \\ \varepsilon & : 2 \end{cases}$$

$\tilde{p}$ = homogenous deg 0 potential on $\tilde{Q}_f$,

$$\tilde{\phi} = \operatorname{tr} \tilde{p} : \tilde{\mathfrak{M}}(w) \longrightarrow \mathbb{C}$$

$\boxed{Q:}$ Compute KCA's

$p = \tilde{p}|_{\tilde{Q}}$ $\qquad \phi = \operatorname{tr} p : \tilde{\mathcal{X}} \longrightarrow \mathbb{C}$

KHA $= K_{\mathbb{C}^*}(\tilde{\mathcal{X}}, \phi)$ $\qquad$ (grading $\Rightarrow \mathbb{C}^*$-action on X)

Using Hecke correspondence in $\widetilde{\mathfrak{M}}(w) \times \widetilde{\mathfrak{M}}(w)$:

Thm (V-V) There are algebra homomorphisms :

$$K_{\mathbb{C}^*}(\widetilde{\mathfrak{X}}, \phi),\ K_{\mathbb{C}^*}(\widetilde{\mathfrak{X}}, \phi)^{op} \longrightarrow K_{G_W \times \mathbb{C}^\times}\left(\widetilde{\mathfrak{M}}(W)^2, \widetilde{\phi}^{(2)}\right)_{\widetilde{\mathfrak{Z}}(W)}$$

Expectation: $\exists$ algebra with triangular decomposition

$$K_{\mathbb{C}^\times}(\widetilde{\mathfrak{X}}, \phi) \otimes \underset{\text{subalg}}{\text{Cartan loop}} \otimes K_{\mathbb{C}^\times}(\widetilde{\mathfrak{X}}, \phi)^{op} = U$$

+

algebra homomorphism $U \to K_{G_W \times \mathbb{C}^\times}\left(\widetilde{\mathfrak{M}}(W)^2, \widetilde{\phi}^{(2)}\right)_{\widetilde{\mathfrak{Z}}(W)}$

An example

$$p = \varepsilon[\alpha,\alpha^*] \in \mathbb{C}\tilde{Q} \quad \Rightarrow \text{KHA} = K_{\mathbb{C}^\times}(\mathfrak{X},\phi) \simeq U_R(Lg)^+$$

<u>choice 1:</u> $\tilde{p} = \varepsilon[\alpha,\alpha^*] + \varepsilon a^* a \in \mathbb{C}\tilde{Q}^f$

$$\text{crit}(\tilde{\phi}) = \{ x \in \tilde{\mathfrak{M}}(w) \mid [x_\alpha, x_{\alpha^*}] + x_{a^*}x_a = 0 = [x_\varepsilon, \alpha] = x_\varepsilon x_{a^*} = x_a x_\varepsilon \}$$

$\operatorname{Im} x_\varepsilon \subseteq \operatorname{Ker} x_a$; stability $\Rightarrow x_\varepsilon = 0$

$$\Rightarrow \text{crit}(\tilde{\phi}) = \mathfrak{M}(w)$$

Moreover $\text{KCA} = K_{G_w \times \mathbb{C}^\times}(\tilde{\mathfrak{M}}(w)^{(2)}, \tilde{\phi}^{(2)}) \simeq K^{G_w \times \mathbb{C}^\times}(Z(w))$

$\rightsquigarrow$ get usual quantum loop gp

<u>choice 2</u> : $\tilde{p} = \varepsilon[\alpha,\alpha^*] \in \mathbb{C}\tilde{Q}^f$

$\rightsquigarrow$ get shifted quantum loop gps $U_q^{-w} Lg$ $\quad w = \dim W$

Finkelberg–Tsymbaliuk ('19) :

shifted q–loop group $\mathcal{U}_F^{\lambda_+,\lambda_-}(L\mathfrak{g})$ $\lambda_+,\lambda_- \in \mathbb{Z}I$

$\boxed{\text{gens :}}$ $x^{\pm}_{i,m}$, $\psi^{\pm}_{i,n}$, $h_{i,r}$ $r,m,n \in \mathbb{Z}$ $n \geq -\lambda_i^{\pm}$ $r \neq 0$

$\boxed{\text{rels :}}$ as in usual q-loop gp except that

$$\psi_i^{\pm}(u) = \sum_{n \geq -\lambda_i^{\pm}} \psi^{\pm}_{i,\pm n} u^{\mp n} = \psi_{i,\mp\lambda_i^{\pm}} u^{\pm\lambda_i^{\pm}} \exp\left(\pm(q-q^{-1})\sum_{r>0} h_{i,\pm r} z^{\mp r}\right)$$

and $\psi^{\pm \; \pm 1}_{i,\mp\lambda_i^{\pm}}$ invertible

it depends only on $\lambda_+ + \lambda_-$

$\boxed{\begin{array}{l} \textbf{Thm (V-V)} \quad w \in \mathbb{N}I \quad w = \dim W \\ \text{① We have an algebra homomorphism} \\ \qquad \mathcal{U}_F^{0,-w}(L\mathfrak{g}) \longrightarrow K_{G_w \times \mathbb{C}^*}\left(\widetilde{\mathfrak{M}}(w)^2, \tilde{\phi}^{(2)}\right)^R_{\tilde{Z}(w)} \otimes F \\ \text{② It is compatible with integral forms} \end{array}}$

4) KCA's and REPRESENTATIONS, I

grading on $\tilde{Q}_{f,1}$ $\rightsquigarrow$ graded quiver $\tilde{Q}^\bullet_f$

$$\tilde{Q}^\bullet_f \begin{cases} I^\bullet = I \times \mathbb{Z}, \\ \tilde{Q}^\bullet_{f,1} \subseteq \tilde{Q}_{f,1} \times \mathbb{Z} \qquad (h,k) : (s(h),k) \longrightarrow (t(h), k+\deg(h)) \end{cases}$$

$$\rightsquigarrow A = \{ (\sigma(z) = \bigoplus_{(i,k)\in I^\bullet} z^k \, \mathrm{id}_{W_{i,k}}, z) \mid z \in \mathbb{C}^* \} \subseteq G_W \times \mathbb{C}^\times$$

general construction with

$$\tilde{\mathfrak{M}}(w) \qquad A \qquad \tilde{\phi} = \mathrm{tr}(\tilde{\rho} = \varepsilon[a, a^*])$$

+ previous thm

$$\rightsquigarrow \boxed{\begin{array}{c} U_F^{0,-\omega}(Lg) \curvearrowright K_A(\tilde{\mathfrak{M}}(\omega), \tilde{\phi}) \otimes_{\mathcal{Z}} F \\ \wr\wr \\ K(\tilde{\mathfrak{M}}^\bullet(w^\bullet), \tilde{\phi}^\bullet) \otimes_{\mathcal{Z}^\bullet} F \end{array}}$$

$$\tilde{\mathfrak{M}}^A = \tilde{\mathfrak{M}}^\bullet = \text{graded quiver variety}$$

[H'21]: $w = \sum_{i \in I} w_i \delta_i$ $\exists$ subcategory $\mathcal{O}^{0,-w}$ of $\mathcal{U}_F^{0,-w}(L\mathfrak{g})$-mod

$$\{\text{simples of } \mathcal{O}^{0,-w}\}$$

$$\updownarrow$$

$\{L(\Psi) \mid \Psi = (\Psi_i(u))_{i \in I},\ \Psi_i(u)$ rational function of degree $-w_i$ regular at 0 $\}$

Fix $w^\bullet = \sum w_{i,k} \in \mathbb{N} I^\bullet$ s.t. $\sum_k w_{i,k} = w_i$ $\forall i$

Let $L^-(w^\bullet)$ = simple in $\mathcal{O}^{0,-w}$ with loop highest weight

$$\Psi^-_{w^\bullet}(u) = \left(\prod_{k \in \mathbb{Z}} \left(1 - \frac{\mathfrak{z}^k}{u}\right)^{-w_{i,k}} \right)_{i \in I}$$

Thm (V-V) The $\mathcal{U}_{\mathfrak{z}}^{0,-w}(L\mathfrak{g})$-modules

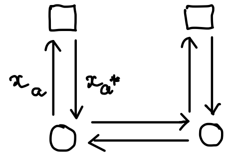
$K(\tilde{m}^\bullet(w^\bullet), \tilde{\phi}^\bullet)$ and $K(\tilde{m}^\bullet(w^\bullet), \tilde{\phi}^\bullet)_{\tilde{\mathcal{L}}^\bullet(w)}$

are both isomorphic to $L^-(w^\bullet)$

Rmk: Henri Liu: a related construction using quasimaps

5) KCA's and REPRESENTATIONS, II

$\mathcal{M}(W)$, Hamiltonian G_W-action

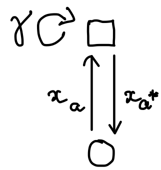


$$\mu_W : \mathcal{M}(W) \longrightarrow \mathfrak{g}_W^\vee$$
$$x \longmapsto x_a x_{a^*}$$

Fix $\begin{cases} \gamma \in \mathfrak{g}_W \text{ nilpotent} & \mathfrak{g}_W = \mathrm{Lie}(G_W) \\ \sigma : \mathbb{C}^\times \longrightarrow G_W \text{ cocharacter s.t. } & \mathrm{Ad}_{\sigma(z)}(\gamma) = z^2 \gamma \end{cases}$

$$\rightsquigarrow \quad A := \{(\sigma(z), z) \mid z \in \mathbb{C}^\times\} \subseteq G_W \times \mathbb{C}^\times$$

$$\phi_\gamma : \mathcal{M}(W) \longrightarrow \mathbb{C}, \qquad x \longmapsto \mathrm{tr}(\gamma x_a x_{a^*})$$



$$\mathcal{M}(W),\ A,\ \phi_\gamma$$

$$\rightsquigarrow \quad \boxed{K_A(\mathcal{M}(W)^2, \phi_\gamma^{(2)})_{\mathcal{Z}(W)} \text{ algebra } \curvearrowright K_A(\mathcal{M}(W), \phi_\gamma)_{\mathcal{L}(W)},\ K_A(\mathcal{M}(W), \phi_\gamma)}$$

$+$ algebra homomorphisms:

$$\boxed{U_F(L\mathfrak{g}) \xrightarrow[{[N]}]{} K^A(\mathcal{Z}(W)) \otimes_R F \longrightarrow K_A(\mathcal{M}(W)^2, \phi_\gamma^{(2)})_{\mathcal{Z}(W)} \otimes_R F}$$

Thm (V–V) Let $w = \ell \delta_i$ and for some $k \in \mathbb{Z}$

$$\sigma(z) = \mathrm{diag}\left(z^{k-\ell+1}, z^{k-\ell+3}, \dots, z^{k+\ell-1}\right)_{W_i}$$

$\gamma \in \mathfrak{gl}(w)$ regular nilpotent. Then

$$K_A(\mathcal{M}(w), \phi_\gamma) \otimes_R F \simeq K_A(\mathcal{M}(w), \phi_\gamma)_{\mathcal{L}(w)} \otimes_R F \simeq KR^{\ell}_{i,k}$$

as $U_\varepsilon(L\mathfrak{g})$ modules

$KR^{\ell}_{i,k}$ = simple module with Drinfeld polynomials $(P_j)_{j \in I}$

$$\begin{cases} P_j(u) = 1 & \forall j \neq i \\ P_i(u) = (1 - q^{k-\ell+1}u)(1-q^{k-\ell+3}u) \dots (1-q^{k+\ell-1}u) \end{cases}$$

$\square$

Remarks

① as before

$$K(\mathcal{M}^\bullet(w), \phi^\bullet_\gamma) \simeq K(\mathcal{M}^\bullet(w), \phi^\bullet_\gamma)_{\mathcal{L}^\bullet(w)} \simeq KR^{\ell}_{i,k}$$

as $U_q(L\mathfrak{g})$ modules

② Nakajima – Okounkov (unpublished) : a similar result

3 can define <u>cohomological critical convolution algebra</u> (using vanishing cycles)

$$\mathcal{U}_3(L\mathfrak{g}) \xrightarrow[{[N]}]{} K(\dot{Z}(W)) \underset{\text{alg}}{\simeq} H_\bullet(\dot{Z}(W), \mathbb{C}) \longrightarrow H^\bullet(\mathfrak{M}^{\bullet 2}(W), \phi^{\bullet(2)}_\gamma)_{\dot{Z}(W)}$$

$$\Rightarrow \mathcal{U}_3(L\mathfrak{g}) \curvearrowright H^\bullet(\mathfrak{M}^\bullet(W), \phi^\bullet_\gamma)_{\dot{\mathcal{L}}(W)}$$

$$\tilde{\Pi} = \text{generalized preprojective algebra of } Q = \mathbb{C}\tilde{Q} \Big/ \left([\alpha,\alpha^*], [\varepsilon,\alpha]\right)$$

$$= \overline{\Pi} \otimes \mathbb{C}[\varepsilon]$$

Lemma $\mathrm{Crit}(\phi^\bullet_\gamma) \cap \mathcal{L}^\bullet(W) = Gr^\bullet_{\tilde{\Pi}}(I_\gamma)$

Thm (V-V) $H^\bullet(\mathfrak{M}^\bullet(\omega), \phi^\bullet_\gamma)_{\mathcal{L}^\bullet(\omega)} \simeq H^\bullet(Gr^\bullet_{\tilde{\Pi}}(I_\gamma), \mathcal{E}_\gamma) = KR^{e}_{i,k}$

$\mathcal{E}_\gamma \in D^b(Gr^\bullet_{\tilde{\Pi}}(I_\gamma))$

$\mathcal{E}_\gamma$ = vanishing cycle complex of $\phi^\bullet_\gamma$

(Another) motivation : cluster structures

$\mathfrak{g}$ simple $/\mathbb{C}$ $\underset{[HL'16]}{\rightsquigarrow}$
- $\mathcal{C}^- \underset{\otimes}{\subseteq} U_\varepsilon(L\mathfrak{g})$-fd mod

  (all f.d. simple are in $\mathcal{C}^-$ up to spectral shift)

- $K_0(\mathcal{C}^-)$ ring with cluster structure s.t. KR cluster variable

$\mathcal{R} = \{\text{cluster variables}\} \subseteq \{\text{cluster monomials}\} \underset{[KKOP]}{\subseteq} \{\text{simple modules}\}$

Combinatorics of Cluster theory ([CC], [DWZ]) $\rightsquigarrow$ $\forall$ $L \in \mathcal{R}$ simple of loop h.w. $w^\bullet \in \mathbb{N}I^\bullet$,

$\exists$ $I_L \in \tilde{\Pi}$-mod, graded, s.t.

$$q\text{-ch}(L) = \sum_{v^\bullet \in \mathbb{N}I^\bullet} \chi(\mathrm{Gr}_{\tilde{\Pi}}(v^\bullet, I_L))\, e^{w^\bullet - C v^\bullet}$$

Euler characteristic

$(I_\gamma = I_{KR^e_{i,k}})$

$\boxed{Q:}$ lift this identity to $U_\varepsilon(L\mathfrak{g})$-modules isomorphism

$$L \simeq H^\bullet(\mathrm{Gr}^\bullet_{\tilde{\Pi}}(I_L), \mathcal{E}_L) \qquad \mathcal{E}_L \in D^b(\mathrm{Gr}^\bullet_{\tilde{\Pi}}(I_L))$$

Previous thm $\Rightarrow$ Ok for KR modules

6) SKETCH of a PROOF

Want: $\boxed{K(\widetilde{\mathcal{M}}^\bullet(\omega^\bullet), \widetilde{\phi}^\bullet) \simeq L^-(\omega^\bullet) \quad \text{as } \mathcal{U}_{\mathfrak{z}}^{0,-\omega}(L\mathfrak{g})\text{-mod}}$

To simplify: $L^-(\omega^\bullet)$ = prefundamental module i.e.

$$\omega^\bullet = \delta_{i,k} \qquad \Psi_{\omega^\bullet}(u) = \left(1, \dots, 1, \left(1 - \frac{\mathfrak{z}^k}{u}\right)^{-1}, 1, \dots, 1\right)$$

i

- both are loop h.w. modules of same l.h.w. $\Rightarrow$ enough to prove

$$\forall v \in \mathbb{N}I^\bullet: \quad q\text{-ch}\left(K(\widetilde{\mathcal{M}}^\bullet(v; \omega^\bullet), \widetilde{\phi}^\bullet)\right) = q\text{-ch}\left(L^-(\omega^\bullet)\right)$$

- [Hernandez–Jimbo, '12]:

$$q\text{-ch}(L^-(\omega^\bullet)) = \lim_{\ell \to \infty} q\text{-ch}\left(KR^{\ell}_{i,1+k-\ell}\right)$$

- [VV] $\Rightarrow$ enough to prove: $\forall v \in \mathbb{N}I^\bullet$

$$K(\widetilde{\mathcal{M}}^\bullet(v; \omega^\bullet), \widetilde{\phi}^\bullet) \underset{\text{v.sp.}}{\simeq} \lim_{\ell \to \infty} K\left(\mathcal{M}^\bullet(v; \omega^\bullet(\ell)), \phi^\bullet_\gamma\right)$$

$$\omega^\bullet(\ell) = \delta_{i,2+k-2\ell} + \delta_{i,4+k-2\ell} + \qquad + \delta_{i,k}$$

– different spaces, different potentials

Key tool: Hirano deformed dimensional reduction

Hirano deformed dimensional reduction

X smooth $\qquad E, E^* \in \mathrm{Vect}(X)$

$\pi : E^* \longrightarrow X \qquad s \in \Gamma(X, E)$

$\sigma = (s, -) : \mathrm{Tot}(E^*) \longrightarrow \mathbb{C}$

$\phi : X \longrightarrow \mathbb{C}$

$$\boxed{\sigma + \pi^*\phi \,,\ \phi_{|s^{-1}(0)} \text{ regular} \Rightarrow K(E^*, \sigma + \pi^*\phi) = K(s^{-1}(0), \phi_{|s^{-1}(0)})}$$