Adjustment Matrices for Iwahori-Hecke Algebras and q-Schur Algebras

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Throughout:

- $e \in \mathbb{Z}_{\geq 2}$;
- $\mathbb{F}$ = a field of characteristic p ($p = 0$ or prime) with $p \nmid e$ or $p = e$;
- $q \in \mathbb{F}$: a primitive e-th root of $1_{\mathbb{F}}$ or $q = 1_{\mathbb{F}}$.

Partitions and e-Abacus

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To display $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ on the e-abacus with s beads, where $s \geq \ell$,

1. put s beads at the least s positions of the abacus, i.e. at positions 0 to $s-1$,
2. move the bead at position $s-1$ to $s-1+\lambda_1$,
3. move the bead at position $s-2$ to $s-2+\lambda_2$,
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In this talk, we shall always choose s so that $e \mid s$.

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When all the beads in a given abacus display of λ are moved as high up their respective runners as possible, we obtain the abacus display of its **e-core**. The **e-weight** of λ is the total number of times the beads in the abacus display move one position up their respective runners to obtain its e-core. The **e-quotient** of λ is the e-tuple $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e-1)})$ where $\lambda^{(i)}$ is the partition read off from runner i (treating runner i as a '1-abacus') of the abacus display of λ.

Removable, addable, normal and co-normal beads

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- **removable** if its preceding position is vacant;
- **addable** if its succeeding position is vacant;

Let $i \in \mathbb{Z}/e\mathbb{Z}$. Iteratively pair the removable i-beads (of λ) with the addable $(i-1)$-beads so that if a removable i-bead at position a is paired with an addable $(i-1)$-bead at position b, then $a < b$ and there are no unpaired removable i-beads or addable $(i-1)$-beads between a and b. The removable i-beads and the addable $(i-1)$-beads that are left unpaired are **normal** and **co-normal** respectively.

Write:

$$\varepsilon_i(\lambda) = \text{ number of normal } i\text{-beads of } \lambda;$$
$$\varphi_i(\lambda) = \text{ number of co-normal } (i-1)\text{-beads of } \lambda.$$

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For $k \in \mathbb{Z}^+$,

$\tilde{e}_i^k \lambda =$ partition obtained from λ by moving its k least normal i-beads to their respective preceding positions (when $k \leq \varepsilon_i(\lambda)$),

$\tilde{f}_i^k \lambda =$ partition obtained from λ by moving its k largest co-normal $(i-1)$-beads to their respective succeeding positions (when $k \leq \varphi_i(\lambda)$).

Iwahori-Hecke Algebras and q-Schur algebras

The Iwahori-Hecke algebra $\mathscr{H}_n^p = \mathscr{H}_{\mathbb{F},q}(n)$, is the $\mathbb{F}$-algebra generated by $\{T_1, \dots, T_{n-1}\}$ subject to:

$$\begin{aligned} T_i^2 &= (q-1)T_i + q; \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}; \\ T_iT_j &= T_jT_i \quad (|j-i| \geq 2). \end{aligned}$$

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The q-Schur algebra $\mathscr{S}_n^p = \mathscr{S}_{\mathbb{F},q}(n)$ is a quasi-hereditary cover of $\mathscr{H}_n^p$.

Specht modules and Weyl modules

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The Specht module S_p^λ has a simple head D_p^λ when λ is e-regular, and $\{D_p^\lambda : \lambda \vdash_{\mathsf{reg}} n\}$ is a complete set of non-isomorphic simple $\mathscr{H}_n^p$-modules.

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The Schur functor f has the following effect on these modules:

$$f(\Delta_p^\lambda) = S_p^\lambda,$$
$$f(L_p^\lambda) = \begin{cases} D_p^\lambda, & \text{if } \lambda \text{ is } e\text{-regular;} \\ 0, & \text{otherwise.} \end{cases}$$

Decomposition Matrices

Let $d^p_{\lambda\mu} = [\Delta^\lambda_p : L^\mu_p]$. Then $d^p_{\lambda\mu} = [S^\lambda_p : D^\mu_p]$ when μ is e-regular.

Collecting this integers in a matrix, we obtain the decomposition matrices

$$\mathbf{D}^{\mathcal{S}}_p := (d^p_{\lambda\mu})_{\lambda,\mu\vdash n}, \qquad \mathbf{D}^{\mathscr{H}}_p = (d^p_{\lambda\mu})_{\lambda\vdash n,\, \mu\vdash_{\mathsf{reg}} n}$$

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for $\mathcal{S}_n$ and $\mathscr{H}_n$ respectively.

Some elementary properties of $d^p_{\lambda\mu}$:

- $d^p_{\lambda\mu} \neq 0$ only if λ and μ have the same e-core, the same e-weight and $\lambda \trianglelefteq \mu$.
- $d^p_{\lambda\lambda} = 1$.

Adjustment Matrices

There is a $\mathbb{Z}$-module homomorphism $\mathsf{d}_p^{\mathcal{S}} : K(\mathcal{S}_n^0) \to K(\mathcal{S}_n^p)$, satisfying

$$\mathsf{d}_p^{\mathcal{S}}([\Delta_0^\lambda]) = [\Delta_p^\lambda];$$
$$\mathsf{d}_p^{\mathcal{S}}([L_0^\lambda]) = \sum_{\mu \vdash n} a_{\lambda\mu}^{\mathcal{S}} [L_p^\mu]$$

with $a_{\lambda\mu}^{\mathcal{S}} \in \mathbb{Z}_{\geq 0}$. The matrix $\mathbf{A}_p^{\mathcal{S}} = (a_{\lambda\mu}^{\mathcal{S}})_{\lambda,\mu\vdash n}$ is the **adjustment matrix** for $\mathcal{S}_n$.

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Similarly, we also have $\mathsf{d}_p^{\mathcal{H}} : K(\mathcal{H}_n^0) \to K(\mathcal{H}_n^p)$ satisfying $\mathsf{d}_p^{\mathcal{H}}([S_0^\lambda]) = [S_p^\lambda]$ and $\mathsf{d}_p^{\mathcal{H}}([D_0^\lambda]) = \sum_{\mu\vdash n} a_{\lambda\mu}^{\mathcal{H}} [D_p^\mu]$ with $a_{\lambda\mu}^{\mathcal{S}} \in \mathbb{Z}_{\geq 0}$, giving rise to the adjustment matrix $\mathbf{A}_p^{\mathcal{H}} = (a_{\lambda\mu}^{\mathcal{H}})_{\lambda,\mu\vdash_{\mathsf{reg}}\, n}$ for $\mathcal{H}_n$.

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By considering the Schur functor, we have $a_{\lambda\mu}^{\mathcal{S}} = a_{\lambda\mu}^{\mathcal{H}}$ when λ and μ are e-regular, so that $\mathbf{A}_p^{\mathcal{H}}$ is a submatrix of $\mathbf{A}_p^{\mathcal{S}}$ and we may write $a_{\lambda\mu}$ without ambiguity.

Relationship between Decomposition and Adjustment Matrices

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Immediate consequence:

- $a_{\lambda\mu} \neq 0$ only if λ and μ has the same e-core, the same e-weight and $\lambda \trianglelefteq \mu$.
- $a_{\lambda\lambda} = 1$ for all $\lambda \vdash n$.
- If $d^p_{\lambda\mu} = d^0_{\lambda\mu}$, then $a_{\nu\mu} = 0$ for all $\nu \neq \mu$ with $d^0_{\lambda\nu} \neq 0$; in particular, $a_{\lambda\mu} = \delta_{\lambda\mu}$.

i-Restriction and i-Induction Functors

Let $i \in \mathbb{Z}/e\mathbb{Z}$ and $k \in \mathbb{Z}^+$. We have induction and restriction functors $E_i^{(k)}, F_i^{(k)} : \bigoplus_n \text{mod-}\mathcal{S}_n^p \to \bigoplus_n \text{mod-}\mathcal{S}_n^p$ satisfying:

$$[E_i^{(k)}(\Delta_p^\lambda)] = \sum_{\check{\lambda}} [\Delta_p^{\check{\lambda}}];$$
$$[F_i^{(k)}(\Delta_p^\lambda)] = \sum_{\hat{\lambda}} [\Delta_p^{\hat{\lambda}}].$$

where the sums run over all partitions $\check{\lambda}$ and $\hat{\lambda}$ obtained from λ by moving k removable i-beads to their respective vacant preceding positions and k addable $(i-1)$-beads to their respective vacant succeeding positions respectively.

Furthermore,

- $E_i^{(k)}(L_p^\lambda)$ has a simple head and a simple socle both isomorphic to $L_p^{\tilde{e}_i^k \lambda}$ if $\varepsilon_i(\lambda) \geq k$;
- $E_i^{(k)}(L_p^\lambda) = L_p^{\tilde{e}_i^k \lambda}$ if $\varepsilon_i(\lambda) = k$;
- $E_i^{(k)}(L_p^\lambda) = 0$ if $\varepsilon_i(\lambda) < k$.

Furthermore,

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- $E_i^{(k)}(L_p^\lambda) = 0$ if $\varepsilon_i(\lambda) < k$.

Similarly,

- $F_i^{(k)}(L_p^\lambda)$ has a simple head and a simple socle both isomorphic to $L_p^{\tilde{f}_i^k \lambda}$ if $\varphi_i(\lambda) \geq k$;
- $F_i^{(k)}(L_p^\lambda) = L_p^{\tilde{f}_i^k \lambda}$ if $\varphi_i(\lambda) = k$;
- $F_i^{(k)}(L_p^\lambda) = 0$ if $\varphi_i(\lambda) < k$.

Rules for $a_{\lambda\mu}$: I

Theorem (Row removal (Fayers, Low))

If $\lambda = (\lambda_1, \lambda_2, \dots)$ *and* $\mu = (\mu_1, \mu_2, \dots)$ *with* $\lambda_1 = \mu_1$. *Then*

$$a_{\lambda\mu} = a_{\lambda^{\geq 2}\mu^{\geq 2}},$$

where $\lambda^{\geq 2} = (\lambda_2, \lambda_3, \dots)$ *and* $\mu^{\geq 2} = (\mu_2, \mu_3, \dots)$.

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Theorem (Column removal (Low))

If $\lambda = (\lambda_1, \dots, \lambda_\ell)$ *and* $\mu = (\mu_1, \dots, \mu_\ell)$ *(with* $\lambda_\ell, \mu_\ell > 0$*). Then*

$$a_{\lambda\mu} = a_{\tilde{\lambda}\tilde{\mu}},$$

where $\tilde{\lambda} = (\lambda_1 - 1, \dots, \lambda_\ell - 1)$ *and* $\tilde{\mu} = (\mu_1 - 1, \dots, \mu_\ell - 1)$.

Rules for $a_{\lambda\mu}$: II

Theorem (Fayers)

$a_{\lambda\mu} = a_{\mathsf{m}(\lambda)\mathsf{m}(\mu)}$ *when* λ *and* μ *are* e*-regular.*

Here, $D^{\mathsf{m}(\nu)}$ is the dual of D^{ν} induced by the algebra involution on $\mathscr{H}_n$ defined by $T_i \mapsto -T_i + q - 1$ for all i.

Rules for $a_{\lambda\mu}$: III

Theorem (Fayers)

Let λ *be a partition with* e*-weight* w*. If*

- $w < p$,
- $a_{\nu\mu} = 0$ *for all* $\lambda \lhd \nu \lhd \mu$,
- $d_{\lambda\mu}(q) \in \{0, q\}$,

then $a_{\lambda\mu} = 0$.

Here $d_{\lambda\mu}(q)$ *is the* q*-decomposition number arising from the Fock space representation of* $U_q(\widehat{\mathfrak{sl}}_e)$.

Rules for $a_{\lambda\mu}$: IV

Theorem (Fayers-T., Low)

- $a_{\lambda\mu} \neq 0$ *only if* $\varepsilon_i(\lambda) \geq \varepsilon_i(\mu)$ *(equivalently,* $\varphi_i(\lambda) \geq \varphi_i(\mu)$*) for all* $i \in \mathbb{Z}/e\mathbb{Z}$.
- *If* $\varepsilon_i(\lambda) = \varepsilon_i(\mu)$ *(equivalently,* $\varphi_i(\lambda) = \varphi_i(\mu)$*), then*

$$a_{\lambda\mu} = a_{\tilde{e}_i^k\lambda,\tilde{e}_i^k\mu} = a_{\tilde{f}_i^l\lambda,\tilde{f}_i^l\mu},$$

where $k = \varepsilon_i(\lambda) = \varepsilon_i(\mu)$ *and* $l = \varphi_i(\lambda) = \varphi_i(\mu)$.

From now on, we fix an e-core partition κ. We consider only partitions with this e-core, and we display all such partitions with s beads, for sufficiently large s such that $e \mid s$.

Denote by r_i the largest occupied position in runner i of κ.

Computed $a_{\lambda\mu}$: Rouquier blocks

Theorem

Let λ and μ be partitions with e-core κ and e-weight w, and e-quotients $(\lambda^{(0)}, \dots, \lambda^{(e-1)})$ and $(\mu^{(0)}, \dots, \mu^{(e-1)})$ respectively.
Suppose that $|r_i - r_j| > (w-1)e$ for all $i, j \in \mathbb{Z}/e\mathbb{Z}$ with $i \neq j$.

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Suppose that $|r_i - r_j| > (w-1)e$ *for all* $i, j \in \mathbb{Z}/e\mathbb{Z}$ *with* $i \neq j$.

1. [Chuang-T., Miyachi, Turner, James-Lyle-Mathas] *If* $p > w$, *then*

$$a_{\lambda\mu} = \delta_{\lambda\mu}.$$

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2. [James-Lyle-Mathas] *If $p \leq w$, then $a_{\lambda\mu} \neq 0$ only if $|\lambda^{(i)}| = |\mu^{(i)}|$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.*
3. [Turner] *If $e = p \leq w$, and λ and μ are e-regular, then*

$$a_{\lambda\mu} = \prod_{i=0}^{e-1} d_{\lambda^{(i)}\mu^{(i)}}.$$

Computed $a_{\lambda\mu}$: Beyond Rouquier blocks

Define a total order $\preceq$ on $\mathbb{Z}/e\mathbb{Z}$ by $i \preceq j$ if and only if $r_i \leq r_j$.

For each $i \in \mathbb{Z}/e\mathbb{Z}$, let i^+ and i^- be respectively the succeeding and preceding elements of i in $\mathbb{Z}/e\mathbb{Z}$ with respect to $\preceq$ (if they exist).

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Theorem (T.)

Let μ be a partition, with e-core κ and e-quotient $(\mu^{(0)}, \dots, \mu^{(e-1)})$. Suppose that:

- $|\mu^{(i^-)}| + |\mu^{(i)}| + |\mu^{(i^+)}| \leq \left\lceil \frac{r_i - r_{i^-}}{e} \right\rceil$ *for all* $i \in \mathbb{Z}/e\mathbb{Z}$*;*
- *whenever* $i \prec j$ *satisfy* $|\mu^{(i^-)}| + |\mu^{(i)}| = \left\lceil \frac{r_i - r_{i^-}}{e} \right\rceil$ *and* $|\mu^{(j)}| + |\mu^{(j+1)}| = \left\lceil \frac{r_j - r_{j^-}}{e} \right\rceil$*, there exists* k *with* $i \prec k \prec j$ *such that* $r_k - r_{k^-} > e$*;*
- $|\mu^{(i)}| < p$ *for all* $i \in \mathbb{Z}/e\mathbb{Z}$*.*

Then $d^p_{\lambda\mu} = d^0_{\lambda\mu}$ *for all* λ*. In particular,* $a_{\lambda\mu} = \delta_{\lambda\mu}$*.*

Computed $a_{\lambda\mu}$: Small e-weights

Theorem (Richards, Fayers, Schroll-T., Low)

Let λ be a partition of e-weight w, with $w \leq 4$ and $w < p$. Then

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Theorem (Fayers, Fayers-T., Low-T.)

Let λ be a partition of e-weight w, with $w \in \{2, 3\}$ and $w \geq p$. Then

$$a_{\lambda\mu} = \delta_{\lambda\mu},$$

except for those 'coming from' the Rouquier blocks.

Theorem (Low)

Let λ, μ be e-regular partitions with empty e-core and e-weight 5, and assume that $e \neq 4$ and $p \geq 5$. Then

$$a_{\lambda\mu} = \delta_{\lambda\mu}.$$

Computed $d_{\lambda\mu}$

Theorem (Kleshchev, T.-Teo, Chuang-T.)

Let λ be a partition and let μ be a partition obtained from λ by moving some removable beads to their respective vacant preceding positions and some addable beads to their respective vacant succeeding positions. Assume that the removable beads moved do not lie in adjacent runners, and that $e = p$. Then

$$d^p_{\lambda\mu} = d^0_{\lambda\mu}.$$

In particular, $a_{\lambda\mu} = \delta_{\lambda\mu}$.

Conjecture

Let λ be a partition, with e-core κ and e-quotient $(\lambda^{(0)}, \dots, \lambda^{(e-1)})$. If

- $|\lambda^{(i)}| \leq 1$ *for all* $i \in \mathbb{Z}/e\mathbb{Z}$;
- *whenever* $\lambda^{(i)}$ *and* $\lambda^{(j)}$ *are nonempty (and* $i \neq j$*), there exists* $k \in \mathbb{Z}/e\mathbb{Z}$ *such that* r_k *is between* r_i *and* r_j*,*

then

$$a_{\lambda\mu} = \delta_{\lambda\mu}.$$

Thank you for your attention!