Semisimple Specht modules indexed by bihooks

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1 Decomposable Specht modules in level 1

Let $\mathbb{F}$ be a field of characteristic $p \geqslant 0$ throughout.

The Specht modules $\{\mathrm{S}_\lambda \mid \lambda \vdash n\}$ over $\mathfrak{S}_n$ are the ordinary irreducible $\mathfrak{S}_n$-modules, indexed by partitions λ of n.

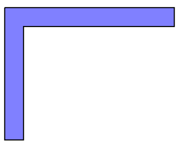
We have the following fundamental fact about Specht modules.

Theorem 1.1 [**Jam78, Corollary 13.18**]. *If $p \neq 2$ or λ is 2-regular, then S_λ is indecomposable.*

When $p = 2$ and λ is 2-singular, it is a difficult problem to determine whether or not S_λ is decomposable. However, some special cases are very tractable.

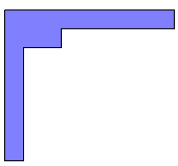
Theorem 1.2 [**Mur80, Theorems 4.1 and 4.5**]. *Let $\lambda = (a, 1^b)$, with $a + b = n$. If n is even, then S_λ is indecomposable.*

If n is odd and $n \geqslant 2b$, then S_λ is indecomposable if and only if $a-b-1 \equiv 0 \pmod{2^L}$, where $2^{L-1} \leqslant b < 2^L$.



Given that S_λ is decomposable if and only if $\mathrm{S}_{\lambda'}$ is, where λ' is the conjugate of λ, the restriction that $a \geqslant b$ is in fact not a problem, and Murphy's result gives a complete classification of which Specht modules indexed by hook partitions are decomposable.

Some further work on this problem includes results of Dodge and Fayers in which they found a new family of decomposable Specht modules indexed by partitions of the form $\lambda = (a, 3, 1^b)$ (subject to some conditions) in [DF12, Theorem 3.1].



A natural generalisation of this problem is to instead consider Specht modules over the Iwahori–Hecke algebra of the symmetric group. This is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$(T_i - q)(T_i + 1) = 0 \qquad \text{for all } i,$$

$$T_iT_j = T_jT_i \qquad \text{for } |i-j|>1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad \text{for } 0 \leqslant i \leqslant n-2,$$

where $q \in \mathbb{F}$ is a primitive eth root of unity.

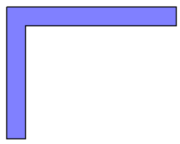
Now the Specht modules $\{\mathrm{S}_\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions λ of n.

As for symmetric groups, we have (following [DJ87, Theorem 3.5]) that S_λ is decomposable if and only if $\mathrm{S}_{\lambda'}$ is and:

Theorem 1.3 [DJ91, Corollary 8.7]. *If $e \neq 2$ or λ is 2-regular, then S_λ is indecomposable.*

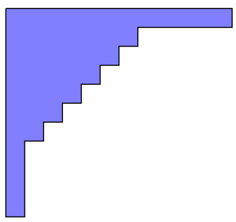
Once again, when $e = 2$ (i.e. $q = -1$), and λ is 2-singular, it is difficult to determine whether or not S_λ is decomposable. Some cases have been solved. First, we have extended Murphy's results on hooks to $\mathscr{H}_n$.

Theorem 1.4 [Spe14, Theorem 6.12]. *Suppose $p \neq 2$ and $\lambda = (a, 1^b)$. Then S_λ is indecomposable if and only if n is even or $b = 2$ or 3 with $p \mid \lceil \frac{a}{2} \rceil$.*

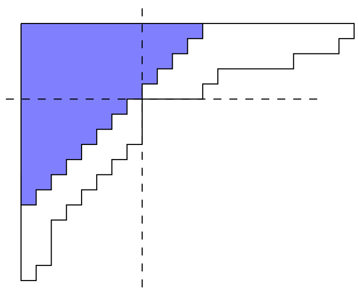


Further classifications:

Theorem 1.5 [DG20, Theorem 6.2]. *For $\lambda = (a, m, m-1, m-2, \dots, 2, 1^b)$, the decomposition of S_λ into indecomposable summands (as Young modules) is given.*



Theorem 1.6 [BBS19, Theorem 6.2]. *Over $\mathbb{C}$, if λ is 2-separated, then S_λ is semisimple, and all composition factors are given.*



2 KLR algebras

We may further generalise our setting to cyclotomic Hecke algebras, deformations of the complex reflection groups $G(l, 1, n) = \mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n$. For our purposes, the following theorem of Brundan and Kleshchev will provide the perspective we take in looking for decomposable Specht modules.

Theorem 2.1 [BK09, **Main Theorem**]. *The (integral) cyclotomic Hecke algebra in quantum characteristic $e \geqslant 2$ is isomorphic to a level l cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ of type $A_{e-1}^{(1)}$ if $e < \infty$, or A_∞ if $e = \infty$ (i.e. corresponding to dominant weight $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2} + \cdots + \Lambda_{\kappa_l}$).*

The cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ is a unital, associative $\mathbb{F}$-algebra with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{\psi_1, \psi_2, \ldots \psi_{n-1}\}$$

subject to a long list of relations. This algebra is naturally $\mathbb{Z}$-graded, which leads us to studying the graded representation theory of cyclotomic Hecke algebras.

2.1 Specht modules over $\mathscr{R}_n^\Lambda$

There is a theory of Specht modules over cyclotomic Hecke algebras, which naturally lead to Specht modules over $\mathscr{R}_n^\Lambda$, which are the ordinary irreducibles.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ be an l-multipartition of n and let $\mathtt{T}_\lambda$ denote the *column initial* λ-tableau, and denote by $\mathbf{i}_\lambda$ its residue sequence modulo e.

Example. Let $\lambda = ((4,3),(3,2,1))$, $e = 3$, and $\Lambda = 2\Lambda_0$. Then

$$\mathtt{T}_\lambda = \begin{array}{l} \begin{array}{|c|c|c|c|}\hline 7 & 9 & 11 & 13 \\ \hline 8 & 10 & 12 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \\ \\ \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \end{array} \quad \text{with residue pattern} \quad \begin{array}{l} \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \\ \\ \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \end{array}$$

and $\mathbf{i}^\lambda = (0,2,1,1,0,2,0,2,1,0,2,1,0)$.

Following [KMR12], the Specht module S_λ is the cyclic $\mathscr{R}_n^\Lambda$-module with homogeneous generator z^λ subject to the following relations.

(i) $e(\mathbf{i})z_\lambda = \delta_{\mathbf{i},\mathbf{i}_\lambda} z_\lambda$;

(ii) $y_r z_\lambda = 0$ for all r;

(iii) $\psi_r z_\lambda = 0$ whenever r and $r+1$ are in the same column of $\mathtt{T}_\lambda$;

(iv) Garnir relations.

As in the classical case of the symmetric group, S_λ has a (homogenous) basis indexed by standard λ-tableaux.

Theorem 2.2 ([Rou08, FS16]). *If $e \neq 2$ and $\kappa_i \neq \kappa_j$ for all $i \neq j$, or if λ is a* conjugate Kleshchev multipartition, *then S_λ is indecomposable.*

It is natural to now look for decomposable Specht modules in higher levels.

3 Decomposable Specht modules in level 2

We now fix $l = 2$, so that $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2}$ and $\mathscr{R}_n^\Lambda$ is isomorphic to a Hecke algebra of type B. We study Specht modules indexed by *bihooks* $\lambda = ((a, 1^b), (c, 1^d))$, a natural generalisation of hooks in level 1. For simplicity, we focus on the case where $e \neq 2$ and $\kappa_1 = \kappa_2$ (WLOG we can assume that $\kappa = (0, 0)$), though most of our results extend to the $e = 2$ case.

In previous work, we found large families of decomposable Specht modules indexed by bihooks.

- [SS20, Theorem 3.8] If $n < 2e$, then S_λ is indecomposable (easy to see that all such bihooks are 'conjugate Kleshchev'). If $n = 2e$, S_λ is decomposable if and only if $p \neq 2$, and $\lambda = ((a, 1^b), (a, 1^b))$ for some a, b (proved by an endomorphism computation).

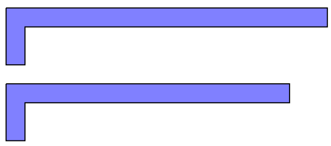
- If $k, j \geqslant 1$, and $\lambda = ((ke), (je))$, then:

  (i) if $j = 1$ or $k = 1$, then S_λ is decomposable if and only if $p \nmid j + k$.

  (ii) if $j, k > 1$, and $j + k$ is even and $p \neq 2$, or if $j + k$ is odd, then S_λ is decomposable.

- Using Brundan and Kleshchev's i-induction and i-restriction functors, we can extend the results from $\lambda = ((ke), (je))$ to $\lambda = ((ke + a, 1^b), (je + a, 1^b))$, for any $0 < a \leqslant e$ and $0 \leqslant b < e$ with $a + b \neq e$, or for $a = b = 0$.

  These bihooks have short legs, for example

  $[\lambda] =$ 

- We conjectured that when $e \neq 2$ and $p \neq 2$, the above (+ their conjugates) provide a complete list of decomposable Specht modules indexed by bihooks.

Now, we want to understand the structure of these (usually) decomposable Specht modules. In order to talk about this, we briefly recall that the simple $\mathscr{R}_n^\Lambda$-modules can be obtained (up to grading shifts) as heads D_λ of Specht modules S_λ indexed by bipartitions λ that are *conjugate-Kleshchev*. We do not define this condition here, as it will not be used in the talk.

Lemma 3.1 ([MSS20]). *Let $k \geqslant j \geqslant 1$. Then $\mathrm{S}_{((je),(ke))} \cong \mathrm{S}^{\circledast}_{((ke),(je))}\langle j + k\rangle$ as graded $\mathscr{R}_n^\Lambda$-modules.*

We can now assume that $k \geqslant j$ without loss of generality. The following similar result will also be useful to us.

Proposition 3.2 ([MSS20]). *Let $k \geqslant j \geqslant 1$. Then $\mathrm{S}_{((ke),(je))} \cong \mathrm{S}^{\circledast}_{((ke),(je))}\langle -2j\rangle$ as graded $\mathscr{R}_n^\Lambda$-modules. Further, the direct summands of $\mathrm{S}_{((ke),(je))}$ are self-dual, up to a grading shift.*

Example. Our running examples for the rest of the talk will be $\mathrm{S}_{((4e),(e))}$ and $\mathrm{S}_{((8e),(3e))}$ (choose your favourite e – it doesn't matter).

We take a brief detour to recall that the Schur algebra $S(n,n)$ is the quasi-hereditary cover of $\mathfrak{S}_n$. Its standard/cell modules $\Delta(\lambda)$ are indexed by partitions λ of n, and each has a simple head denoted $L(\lambda)$. The Schur functor (a functor from $S(n,n)$-Mod to $\mathfrak{S}_n$-Mod sends $\Delta(\lambda)$ to $(\mathrm{S}_\lambda)^*$, and $L(\lambda)$ to $\hat{\mathrm{D}}_\lambda$ if λ is p-restricted, and 0 otherwise. Here $\hat{\mathrm{D}}_\lambda$ denotes the simple head of the dual Specht module $(\mathrm{S}_\lambda)^*$, or in other words the simple socle of S_λ. If μ is p-restricted, then $[\Delta(\lambda) : L(\mu)] = [(\mathrm{S}_\lambda)^* : \hat{\mathrm{D}}_\mu]$.

Thanks to work of James, these decomposition numbers are known (given by an explicit formula) when λ is a two-column partition.

Theorem 3.3 ([KM17, MSS20])**.** *There is a Morita equivalence between $S(n,n)$ and a certain quotient $\mathscr{S}_n$ of $\mathscr{R}_n^\Lambda$. The modules $\mathrm{S}_{((ke),(je))}$ factor through the quotient, and are mapped to $\Delta(1^j)\otimes\Delta(1^k) \cong \Delta(1^k)\otimes\Delta(1^j)$ under this Morita equivalence.*

Remark. The grading is lost under the above Morita equivalence.

The dual Pieri rule now tells us that $\Delta(1^k)\otimes\Delta(1^j)$ has a filtration by Weyl modules $\Delta(1^{k+j}), \Delta(2,1^{k+j-2}), \Delta(2^2,1^{k+j-4}), \dots, \Delta(2^j,1^{k-j})$. Each such Weyl module occurs exactly once in this filtration.

Now, we also know that Weyl modules are indecomposable. It follows that $\mathrm{S}_{((ke),(je))}$ can only possibly be semisimple if each of the modules $\Delta(1^{k+j})$, $\Delta(2,1^{k+j-2})$, $\Delta(2^2,1^{k+j-4})$, $\dots$, $\Delta(2^j,1^{k-j})$ is irreducible. There is a nice combinatorial criterion to determine whether a given $\Delta(\lambda)$ is irreducible. However, since we need several to be simultaneously irreducible, it turns out that the criterion implies an even simpler characterisation.

Theorem 3.4 ([MSS20])**.** *Suppose $p \neq 2$ and $k \geqslant j$. Then $\Delta(1^{k+j})$, $\Delta(2,1^{k+j-2}), \dots,$ $\Delta(2^j,1^{k-j})$ are simultaneously irreducible if and only if p does not divide any of $k+j, k+j-1, \dots, k-j+2$.*

Example.

- The Morita equivalence sends $\mathrm{S}_{((4e),(e))}$ to $\Delta(1^4)\otimes\Delta(1)$, which is filtered by $\Delta(1^5)$ and $\Delta(2,1^3)$. We always have that $\Delta(1^5) \cong L(1^5)$. $\Delta(2,1^3)$ is irreducible (and thus equal to $L(2,1^3)$ if and only if $p \neq 5$. If $p=5$, then $\Delta(2,1^3)$ has simple head $L(2,1^3)$ with simple submodule (i.e. socle) $\Delta(1^5)$.

- It sends $\mathrm{S}_{((8e),(3e))}$ to $\Delta(1^8)\otimes\Delta(1^3)$, which is filtered by $\Delta(1^{11})$, $\Delta(2,1^9)$, $\Delta(2^2,1^7)$, and $\Delta(2^3,1^5)$. If p does not divide any of $11, 10, \dots, 7$, then each of those Weyl modules is irreducible.

  Suppose that $p=3$, so that it divides precisely one of these numbers. In this case, $\Delta(1^{11})$ and $\Delta(2,1^9)$ are still irreducible, but now $\Delta(2^2,1^7)$ has submodule $L(2,1^9)$, and $\Delta(2^3,1^5)$ has submodule $L(1^{11})$. In particular, $\mathrm{S}_{((8e),(3e))}$ has no hope of being semisimple in characteristic 3.

Next, we notice that the *Schur functor* maps $\Delta(1^k)\otimes\Delta(1^j)$ to the signed Young permutation module $M(k,j)\otimes\mathrm{sgn}$. In particular, $\Delta(1^k)\otimes\Delta(1^j)$ decomposes as a direct sum of *listing modules*, which the Schur functor maps to signed Young modules. Essentially, decomposing $M(k,j)$ into Young modules tells us the number of summands of $\mathrm{S}_{((ke),(je))}$! But, even better, it tells us that each summand of $\mathrm{S}_{((ke),(je))}$ is self-dual (up to some grading shift).

Corollary 3.5 ([MSS20]). *Suppose $e, p \neq 2$ and $k \geqslant j$. Then $\mathrm{S}_{((ke),(je))}$ is semisimple if and only if p does not divide any of $k+j, k+j-1, \dots, k-j+2$.*

Remark. In this semisimple situation, we have that

$$\Delta(1^k) \otimes \Delta(1^j) \cong L(1^{k+j}) \oplus L(2, 1^{k+j-2}) \oplus \cdots \oplus L(2^j, 1^{k-j}),$$

but it is not yet clear what these simple modules correspond to on the other side of the Morita equivalence with $\mathscr{S}_n$, nor if there should be any grading shifts present there.

Example.

- When $p \neq 5$, we know that $\mathrm{S}_{((4e),(e))}$ is a direct sum of two simples. What are they? There is a (degree 1) homomorphism $\alpha : \mathrm{S}_{((5e),\varnothing)} \to \mathrm{S}_{((4e),(e))}$, which tells us that this one-dimensional is a submodule. There is another (degree 1) homomorphism $\gamma : \mathrm{S}_{((4e,1),(e-1))} \to \mathrm{S}_{((4e),(e))}$, which tells us that $\mathrm{D}_{((4e,1),(e-1))}$ is a composition factor of $\mathrm{S}_{((4e),(e))}$. It follows that

  $$\mathrm{S}_{((4e),(e))} \cong \mathrm{D}_{((5e),\varnothing)}\langle 1\rangle \oplus \mathrm{D}_{((4e,1),(e-1))}\langle 1\rangle$$

  When $p = 5$, it is not semisimple – in fact our previous result from [SS20] told us it was indecomposable. Since it must be self-dual, we have that $\Delta(1^4) \otimes \Delta(1) \cong L(1^5)|L(2,1^3)|L(1^5)$. Using the homomorphisms α and γ above, we deduce that

  $$\mathrm{S}_{((4e),(e))} \cong \mathrm{D}_{((5e),\varnothing)}\langle 1\rangle | \oplus \mathrm{D}_{((4e,1),(e-1))}\langle 1\rangle | \, \mathrm{D}_{((5e),\varnothing)}\langle 1\rangle.$$

- When $p > 11$, $\mathrm{S}_{((8e),(3e))}$ is semisimple – a direct sum of 4 simples. Figuring out which takes a little extra trick in this example.

  $$\Delta(1^9) \otimes \Delta(1^2) \cong L(1^{11}) \oplus L(2, 1^9) \oplus L(2^2, 1^7),$$

  while

  $$\Delta(1^8) \otimes \Delta(1^3) \cong L(1^{11}) \oplus L(2,1^9) \oplus L(2^2,1^7) \oplus L(2^3,1^5) \cong (\Delta(1^9) \otimes \Delta(1^2)) \oplus L(2^3,1^5),$$

  etc. So we can argue inductively like this, and what we get is that this 'final new summand' $L(2^3, 1^5)$ corresponds to $\mathrm{D}_{((8e,2e+1),(e-1))}$ coming from the (degree 3) homomorphism $\gamma : \mathrm{S}_{((8e,2e+1),(e-1))} \to \mathrm{S}_{((8e),(3e))}$, so we know that $\mathrm{D}_{((8e,2e+1),(e-1))}\langle 3\rangle$ is one of the summands. In fact,

  $$\mathrm{S}_{((8e),(3e))} \cong \mathrm{D}_{((11e),\varnothing)}\langle 3\rangle \oplus \mathrm{D}_{((10e,1),(e-1))}\langle 3\rangle \oplus \mathrm{D}_{((9e,e+1),(e-1))}\langle 3\rangle \oplus \mathrm{D}_{((8e,2e+1),(e-1))}\langle 3\rangle.$$

  Now suppose that $p = 3$. In this case, $\Delta(1^{11})$ and $\Delta(2, 1^9)$ are still irreducible, but now $\Delta(2^2, 1^7)$ has submodule $L(2, 1^9)$, and $\Delta(2^3, 1^5)$ has submodule $L(1^{11})$. On the Schur algebra side, we must combine modules $L(0)$, $L(1)$, $L(1)|L(2)$, and $L(0)|L(3)$ to construct a module that is decomposable and has self-dual summands. The only way to do this is to have

  $$\Delta(1^8) \otimes \Delta(1^3) \cong L(0)|L(3)|L(0) \; \oplus \; L(1)|L(2)|L(1).$$

  The label match-ups and gradings coincide with how they worked in the semisimple case, and this thus translates to

  $$\mathrm{S}_{((8e),(3e))} \cong \begin{array}{c} (\mathrm{D}_{((11e),\varnothing)}\langle 3\rangle | \, \mathrm{D}_{((8e,2e+1),(e-1))}\langle 3\rangle | \, \mathrm{D}_{((11e),\varnothing)}\langle 3\rangle) \\ \oplus \\ (\mathrm{D}_{((10,1),(e-1))}\langle 3\rangle | \, \mathrm{D}_{((9e,e+1),(e-1))}\langle 3\rangle | \, \mathrm{D}_{((10,1),(e-1))}\langle 3\rangle) \end{array}$$

Indeed the general situation looks similar to the above examples when p divides none of, or exactly one of, the integers $k+j$, $k+j-1$, …, $k-j+2$.

Theorem 3.6 ([MSS20])**.** *Suppose $e, p \neq 2$ and $k \geqslant j \geqslant 1$. If p divides none of $k+j$, $k+j-1$, …, $k-j+2$, then*

$$\mathrm{S}_{((ke),(je))} \cong \mathrm{D}_{((ke+je),\varnothing)}\langle j\rangle \oplus \mathrm{D}_{((ke+je-e,1),(e-1))}\langle j\rangle \oplus \mathrm{D}_{((ke+je-2e,e+1),(e-1))}\langle j\rangle \oplus \cdots \oplus \mathrm{D}_{((ke,je-e+1),(e-1))}\langle j\rangle.$$

If p divides exactly one of those integers, then each summand of $\mathrm{S}_{((ke),(je))}$ is either one of the simples above (including the degree shift by j), or a uniserial module of the form $\mathrm{D}_\mu\langle j\rangle | \mathrm{D}_\nu\langle j\rangle | \mathrm{D}_\mu\langle j\rangle$, for μ and ν among the above bipartitions.

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