Using Degrees of Irreducible Characters to Construct a Supercharacter Theory

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Definition of Supercharacter Theory

(See P. Diaconis and I.M. Isaacs, Supercharacters and Superclasses for Algebra Groups in Transactions of the AMS, 2008)

Let G be a finite group.
Let $Irr(G)$ denote the irreducible characters of G.
Let $Ch(G)$ denote a set of characters of G.
Let $\kappa(G)$ denote a partition of the conjugacy classes of G.

A **supercharacter theory** for a group G is a pair $(Ch(G), \kappa(G))$ such that

1. $1_G \in \mathrm{Ch}(G)$ and $\{1\} \in \kappa(G)$.
2. $|Ch(G)| = |\kappa(G)|$.
3. Each character in $Ch(G)$ is constant on each block of $\kappa(G)$.
4. Two distinct characters in $Ch(G)$ have distinct constituents.

Two Trivial Examples

Example 1: Take $Ch(G) = Irr(G)$ and let $\kappa(G)$ have blocks equal to the conjugacy classes. (The supercharacter theory table is then the usual character table.)

Example 2: Take $Ch(G) = \{1_G, \sum_{\chi \in Irr(G), \chi \neq 1_G} \chi(1)\chi\}$ and $\kappa(G) = \{\{1\}, G - \{1\}\}$. Then we get the following supercharacter theory table:

	$\{1\}$	$\{G - \{1\}\}$
1_G	1	1
ρ_G	$\|G\| - 1$	-1

where ρ_G denotes the regular character of G.

Constructing Supercharacter Theories

Let δ be a partitioning of $Irr(G)$ and let $X \in \delta$. Let

$$\sigma_X = \sum_{\chi \in X} \chi(1)\chi.$$

Attempt to construct a partition, $\kappa(G)$, of the set up conjugacy classes of G such that σ_X is constant on each block of $\kappa(G)$ for all $X \in \delta$.

Another Example

$SL_2(3)$	C_1	C_2	C_3	C_4	C_5	C_6	C_7
χ_0	1	1	1	1	1	1	1
χ_1	1	1	ζ^4	ζ^2	1	ζ^4	ζ^2
χ_2	1	1	ζ^2	ζ^4	1	ζ^2	ζ^4
χ_3	2	-2	-1	-1	0	1	1
χ_4	2	-2	ζ^5	ζ	0	ζ^2	ζ^4
χ_5	2	-2	ζ	ζ^5	0	ζ^4	ζ^2
χ_6	3	3	0	0	-1	0	0

$SL_2(3)$	C_1	C_2	C_5	K
χ_0	1	1	1	1
ψ_1	2	2	2	-1
ψ_2	6	-6	0	0
ψ_3	3	3	-1	0

where $K = C_3 \cup C_4 \cup C_6 \cup C_7$ and $\zeta = e^{\frac{\pi i}{3}}$

A Possible Supercharacter Theory

For $i > 1$, let

$$\psi_i = \sum_{\chi \in Irr(G), \chi(1)=i} \chi(1)\chi$$

Let

$$\psi_1 = \sum_{\chi \in Irr(G), \chi(1)=1, \chi \neq 1_G} \chi$$

Let $Ch(G) = \{1_G, \psi_i\}$
Question: For which families of groups does there exist a partitioning of the conjugacy classes $\kappa(G)$, such that $(Ch(G), \kappa(G))$ is a supercharacter theory?

If such a supercharacter theory exist for a given group G call it a **super degree character theory** for G.

Note: This supercharacter theory applies for all abelian groups.

It also works for the group $SL_2(3)$, as above.

Proposition

If G does not have a super degree character theory then G must have at least eight conjugacy classes.

Proof: Check all 36 groups which have at most seven conjugacy classes. □

$GL_2(3)$ has eight conjugacy classes but does not have a super degree character theory.

Example of a group that does not have a super degree character theory

A_7	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_0	1	1	1	1	1	1	1	1	1
χ_1	6	2	3	-1	0	0	1	-1	-1
χ_2	10	-2	1	1	1	0	0	A	B
χ_3	10	-2	1	1	1	0	0	B	A
χ_4	14	2	2	2	-1	0	-1	0	0
χ_5	14	2	-1	-1	2	0	-1	0	0
χ_6	15	-1	3	-1	0	-1	0	1	1
χ_7	21	1	-3	1	0	-1	1	0	0
χ_8	35	-1	-1	-1	-1	1	0	0	0

where $\zeta = e^{\frac{2\pi i}{7}}$, $A = \zeta^3 + \zeta^5 + \zeta^6$ and $B = \zeta + \zeta^2 + \zeta^4$

Example of a group that does not have a super degree character theory

A_7	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_0	1	1	1	1	1	1	1	1	1
χ_1	6	2	3	-1	0	0	1	-1	-1
$\chi_2 + \chi_3$	20	-4	2	2	2	0	0	-1	-1
$\chi_4 + \chi_5$	28	4	1	1	1	0	-2	0	0
χ_6	15	-1	3	-1	0	-1	0	1	1
χ_7	21	1	-3	1	0	-1	1	0	0
χ_8	35	-1	-1	-1	-1	1	0	0	0

Families of groups that do not have a super degree character theory when they are "large enough".

1. $A_n, n \geq 7$
2. $S_n, n \geq 7$
3. $SL_n(q), q \geq 5, n \geq 2$
4. $GL_n(q), q \geq 3, n \geq 2$
5. $GU_n(q), q \geq 3, n \geq 2$
6. $SU_n(q), q \geq 5, n \geq 2$
7. $Sp_n(2)$

Families of groups that do have a super degree character theory:

1. Abelian Groups
2. Orthogonal Groups
3. $C_m \rtimes C_n$ where either m or n is prime
4. Frobenius Groups

Generic Character Table of $SO_3(q)$, q even

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_q	q	0	1	-1
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm}$	0

for $1 \leq m, j \leq \frac{1}{2}(q-2)$,
$1 \leq n, k \leq \frac{q}{2}$

$\zeta_n = e^{\frac{2\pi i}{n}}$

$SO_3(q)$, q even

Lemma

For $1 \leq j \leq \frac{1}{2}(q-2)$ and $1 \leq k \leq \frac{q}{2}$

$$\sum_{m=1}^{\frac{q-2}{2}} \zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm} = -1$$

and

$$\sum_{n=1}^{\frac{q}{2}} -(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq}) = 1$$

$SO_3(q)$, q even

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_q	q	0	1	-1
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm}$	0

So we have the supercharacter theory table is:

	I	A	B	C
χ_0	1	1	1	1
χ_q	q	0	1	-1
χ_{q-1}	$\frac{q(q-1)}{2}$	$-\frac{q}{2}$	0	1
χ_{q+1}	$\frac{(q+1)(q-2)}{2}$	$\frac{q-2}{2}$	-1	0

$B = \cup B_j$, $C = \cup C_k$

Generic Character Table of $SO_3(q)$, q odd

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_1	1	1	$(-1)^j$	$(-1)^k$
$\chi_{q,1}$	q	0	1	-1
$\chi_{q,2}$	q	1	$(-1)^j$	$(-1)^{k+1}$
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn}+\zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm}+\zeta_{q-1}^{-jm}$	0

for $1 \le m \le \frac{1}{2}(q-3)$,
$1 \le n, j \le \frac{1}{2}(q-1)$,
$1 \le k \le \frac{1}{2}(q+1)$

$SO_3(q)$, q odd

Lemma

Let $1 \leq j \leq \frac{q-1}{2}$ and $1 \leq k \leq \frac{q+1}{2}$.

If j and k are odd then

$$\sum_{m=1}^{\frac{q-3}{2}} \zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm} = 0 \text{ and } \sum_{n=1}^{\frac{q-1}{2}} -(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq}) = 0$$

If j and k are even then

$$\sum_{m=1}^{\frac{q-3}{2}} \zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm} = -2 \text{ and } \sum_{n=1}^{\frac{q-1}{2}} -(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq}) = 2$$

$SO_3(q)$, q odd

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_1	1	1	$(-1)^j$	$(-1)^k$
$\chi_{q,1}$	q	0	1	-1
$\chi_{q,2}$	q	1	$(-1)^j$	$(-1)^{k+1}$
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn} + \zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm} + \zeta_{q-1}^{-jm}$	0

	I	A	B_j j odd	B_j j even	C_k k odd	C_k k even
χ_0	1	1	1	1	1	1
χ_1	1	1	-1	1	-1	1
χ_q	$2q$	1	0	2	0	-2
χ_{q-1}	$\frac{(q-1)^2}{2}$	$-\frac{(q-1)}{2}$	0	0	0	2
χ_{q+1}	$\frac{(q+1)(q-3)}{2}$	$\frac{q-3}{2}$	0	-2	0	0

$SO_3(q)$, q odd

So we have the supercharacter theory table is:

	I	A	$B_j \cup C_k$ j,k odd	B_j j even	C_k k even
χ_0	1	1	1	1	1
χ_1	1	1	-1	1	1
χ_q	$2q$	0	0	2	-2
χ_{q-1}	$\frac{(q-1)^2}{2}$	$-\frac{(q-1)}{2}$	0	0	2
χ_{q+1}	$\frac{(q+1)(q-3)}{2}$	$\frac{q-3}{2}$	0	-2	0

Characters of Semidirect Products of Abelian Groups

Suppose $G = A \rtimes B$ where A and B are both abelian.
Let $\{\alpha_j \mid j = 0, \ldots, n_1 - 1\}$ and $\{\beta_k \mid k = 0, \ldots, n_2 - 1\}$ denote the irreducible characters of A and B respectively.

B acts on the irreducible characters of A by ${}^b\alpha_j(a) = \alpha_j({}^b a)$. Let α_j^o denote the orbit containing α_j of this irreducible characters of A under this action. Let B_j be the subgroup of B that is the kernel of the action of B on α_j^o.

Characters of Semidirect Products of Abelian Groups

Given any pair (α_j^o, χ) where χ is an irreducible character of B_j we get an irreducible character $\pi_{(\alpha_j^o,\chi)}$ of G given by (for all $a \in A, b \in B$):

$$\pi_{(\alpha_j^o,\chi)}(ab) = \begin{cases} 0 & \text{if } b \notin B_j \\ \chi(b) \sum_{\alpha \in \alpha_j^0} \alpha(a) & \text{if } b \in B_j \end{cases} .$$

This gives us all the irreducible characters of G.

$\pi_{(\alpha_j^o,\chi)}$ will be denoted by $\alpha_j^o \cdot \chi$ within character tables

Characters of Semidirect Products of Abelian Groups

Example: $C_{14} \rtimes C_3$

Let $A = C_{14}, B = C_3$

$A = \langle a \rangle, B = \langle b \rangle$

B acts on A by ${}^b a = a^9$.

We can take $\alpha_j(a) = \zeta^j$ where $\zeta = e^{\frac{2\pi i}{14}}$ for $j = 0, 1, \ldots, 13$
and $\beta_k(b) = \eta^k$ where $\eta = e^{\frac{2\pi i}{3}}$ for $k = 0, 1, 2$

Characters of Semidirect Products

Example: $C_{14} \rtimes C_3$

B acts on α_j by ${}^b\alpha_j = \alpha_{9j}$.

So ${}^b\alpha_1 = \alpha_9$, ${}^b\alpha_9 = \alpha_{9^2}, \ldots$
But $\zeta^{9^2} = \zeta^{11}$ and $\zeta^{9^3} = \zeta$.
So this action creates the following orbit of irreducible characters of A: $\{\alpha_1, \alpha_9, \alpha_{11}\}$

Similarly ${}^b\alpha_2 = \alpha_{2.9}$, ${}^b\alpha_{2.9} = \alpha_{2.9^2}, \ldots$
But $\zeta^{2\cdot 9} = \zeta^4$, $\zeta^{2\cdot 9^2} = \zeta^8$, $\zeta^{2\cdot 9^3} = \zeta^2$
So we have the orbit $\{\alpha_2, \alpha_4, \alpha_8\}$

Similarly we have the orbits:
$\{\alpha_3, \alpha_{13}, \alpha_5\}$
$\{\alpha_6, \alpha_{12}, \alpha_{10}\}$
$\{\alpha_7\}$ $\quad ({}^b\alpha_7 = \alpha_{7.9} = \alpha_7)$
$\{\alpha_0\}$

Characters of Semidirect Products

Example: $C_{14} \rtimes C_3$

	1	$[a^2b]$	$[a^2b^2]$	$[ab]$	$[ab^2]$	$[a^7]$	$[a]$	$[a^2]$	$[a^3]$	$[a^6]$
$\alpha_0^o \cdot \beta_0$	1	1	1	1	1	1	1	1	1	1
$\alpha_0^o \cdot \beta_1$	1	η	η^2	η	η^2	1	1	1	1	1
$\alpha_0^o \cdot \beta_2$	1	η^2	η	η^2	η	1	1	1	1	1
$\alpha_7^o \cdot \beta_0$	1	1	1	-1	-1	-1	-1	1	-1	1
$\alpha_7^o \cdot \beta_1$	1	η	η^2	$-\eta$	$-\eta^2$	-1	-1	1	-1	1
$\alpha_7^o \cdot \beta_2$	1	η^2	η	$-\eta^2$	$-\eta$	-1	-1	1	-1	1
$\alpha_1^o \cdot \beta_0$	3	0	0	0	0	-3	$[\zeta]$	$[\zeta^2]$	$[\zeta^3]$	$[\zeta^6]$
$\alpha_2^o \cdot \beta_0$	3	0	0	0	0	3	$[\zeta^2]$	$[\zeta^2]$	$[\zeta^6]$	$[\zeta^6]$
$\alpha_3^o \cdot \beta_0$	3	0	0	0	0	-3	$[\zeta^3]$	$[\zeta^6]$	$[\zeta]$	$[\zeta^2]$
$\alpha_6^o \cdot \beta_0$	3	0	0	0	0	3	$[\zeta^6]$	$[\zeta^6]$	$[\zeta^2]$	$[\zeta^2]$

$[\zeta] = \zeta + \zeta^9 + \zeta^{11}$ $[\zeta^2] = \zeta^2 + \zeta^4 + \zeta^8$ $[\zeta^3] = \zeta^3 + \zeta^5 + \zeta^{13}$
$[\zeta^6] = \zeta^6 + \zeta^{10} + \zeta^{12}$

Characters of Semidirect Products

Example: $C_{14} \rtimes C_3$

	1	$[a^2b]$	$[a^2b^2]$	$[ab]$	$[ab^2]$	$[a^7]$	$[a]$	$[a^2]$	$[a^3]$	$[a^6]$
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	5	-1	-1	-1	-1	-1	-1	5	-1	5
χ_3	12	0	0	0	0	0	0	-2	0	-2

So the supercharacter table is:

	1	$[a^2b] \cup [a^2b^2] \cup [ab] \cup [ab^2] \cup [a^7] \cup [a] \cup [a^3]$	$[a^2] \cup [a^6]$
χ_0	1	1	1
χ_1	5	-1	5
χ_3	12	0	-2

Semidirect Products and Super Degree Character Theory

Proposition

Let $G = C_m \rtimes C_n$ where either m or n is a prime.
For $i > 1$, let

$$\psi_i = \sum_{\chi \in Irr(G), \chi(1)=i} \chi(1)\chi.$$

Let

$$\psi_1 = \sum_{\chi \in Irr(G), \chi(1)=1, \chi \neq 1_G} \chi.$$

Let $Ch(G) = \{1_G, \psi_i\}$.
There exist a partitioning of the conjugacy classes $\kappa(G)$, such that $(Ch(G), \kappa(G))$ is a supercharacter theory.

Proposition

Let $G = C_m \rtimes C_p$ where p is a prime. Then G has a super degree character theory with the following supercharacter table.

	1	K_2	K_3
χ_0	1	1	1
χ_1	$n_1 p - 1$	-1	$n_1 p - 1$
χ_p	$m - n_1$	0	$-n_1$

where n_1 is the number of orbits of irreducible characters of C_m of length 1.

Corollary

Let $G = D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$ then G has a super degree character theory with the following supercharacter table.

n odd	1	$\{s, sr^i\}$	$\{r^i\}$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	$n-1$	0	-1

or

n even	1	$\{s, sr^i, r^{odd}\}$	$\{r^{even}\}$
χ_0	1	1	1
χ_1	3	-1	3
χ_2	$n-2$	0	-2

$(i = 1, \ldots, n-1)$

Proposition

Let $G = C_p \rtimes C_n$ where p is a prime. Then G has a super degree character theory with the following supercharacter table.

	1	K_2	K_3
χ_0	1	1	1
χ_1	$n-1$	-1	$n-1$
χ_{n_1}	$\frac{n(p-1)}{n_1}$	0	$-\frac{n}{n_1}$

where n_1 is the order of the action of C_n on C_p.

Frobenius Groups and Super Degree Character Theory

Proposition

Let G be a Frobenius group (so $G = \mathbf{F}_q \rtimes \mathbf{F}_q^*$ where q is a power of a prime) then G has a super degree character theory with the following supercharacter table.

	1	A	B
χ_0	1	1	1
χ_1	$q-2$	-1	$q-2$
χ_{q-1}	$q-1$	0	-1

Frobenius Groups and Super Degree Character Theory

Generic character table of $\mathbf{F}_q \rtimes \mathbf{F}_q^*$

	1	A_j	B
χ_0	1	1	1
$\chi_{1,j}$	1	$\beta_j(a_j)$	1
χ_{q-1}	$q-1$	0	-1

Where $j = 1, \ldots, q-2$, the β_j are the nontrivial irreducible characters of $\mathbf{F}_q^*$ and a_j is an element in the conjugacy class A_j that is also in the group $\mathbf{F}_q^*$.

Thank you!