The multiplicity freeness of the 3-weight blocks of the Iwahori-Hecke algebra of type B

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OIST

§1: INTRODUCTION

$n \geq 1$, $\mathbb{F}$ field, $q \in \mathbb{F}^*$, e multiplicative order of q in $\mathbb{F}$.
$Q_1, \dots, Q_r \in \mathbb{F}$.

ARIKI-KOIKE ALGEBRA [Ariki-Koike '94, Broué-Malle '93]

↳ Unital associative $\mathbb{F}$-algebra $\mathcal{H}_n$ with:

Generators: $T_0, \dots, T_{n-1}$

Relations:

$$(T_i + q)\cdot(T_i - 1) = 0 \qquad 1 \leq i \leq n-1$$

$$(T_0 - Q_1)\cdot \ldots \cdot (T_0 - Q_r) = 0$$

$$T_i T_j = T_j T_i \qquad |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$$

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$n \geq 1$, $\mathbb{F}$ field, $q \in \mathbb{F}^*$, e multiplicative order of q in $\mathbb{F}$.

$q^{\kappa_1}, \dots, q^{\kappa_r} \in \mathbb{F}$. $(\kappa_1, \dots, \kappa_r) \in \mathbb{Z}_e^r$: multicharge of $\mathcal{H}_n$.

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$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$$

Representation theory of $\mathcal{H}_n$

$\mathcal{H}_n$ is a cellular algebra.

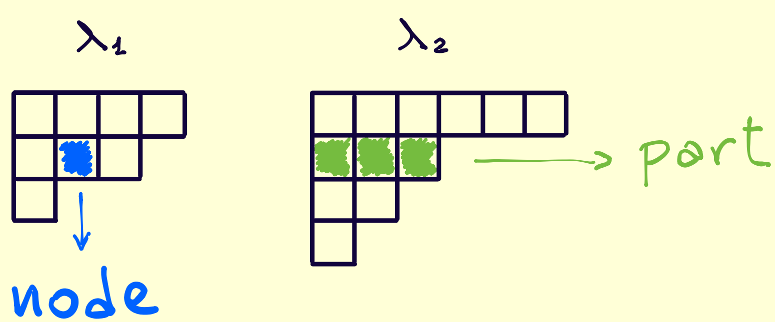
↳ its cellularity is underlined by the poset $(\mathcal{P}(n,r), \trianglelefteq)$:

- $\mathcal{P}(n,r) = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \mid \lambda_i \in \mathcal{P}(x_i) \ \& \ \sum_{i=1}^{r} x_i = n\} =$

  $= \{$ multipartitions of n in r components $\}$;

- $\trianglelefteq$: dominance order on multipartitions.

Example. $n = 20$, $r = 2$. $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = ((4,3,1), (6,3,2,1)) \in \mathcal{P}(20,2)$

$\boldsymbol{\lambda} =$ 

$\lambda \in \mathcal{P}(n,r) \rightsquigarrow S^\lambda$ $\mathcal{H}_n$-module

$\{ S^\lambda \mid \lambda \in \mathcal{P}(n,r) \}$: Specht modules of $\mathcal{H}_n$.

Irreducible $\mathcal{H}_n$-modules

<u>Semisimple case</u>

$\{ S^\lambda \mid \lambda \in \mathcal{P}(n,r) \}$

<u>Non-semisimple case</u>

$\{ D^\lambda \mid \lambda \in \mathcal{P}(n,r)$ & λ is Kleshchev $\}$

$\lambda, \mu \in \mathcal{P}(n,r)$, μ Kleshchev.

Decomposition number : $d_{\lambda\mu} \doteqdot [S^\lambda : D^\mu]$

↳ multiplicity of D^μ as composition factor of S^λ.

Blocks of $\mathcal{H}_n$

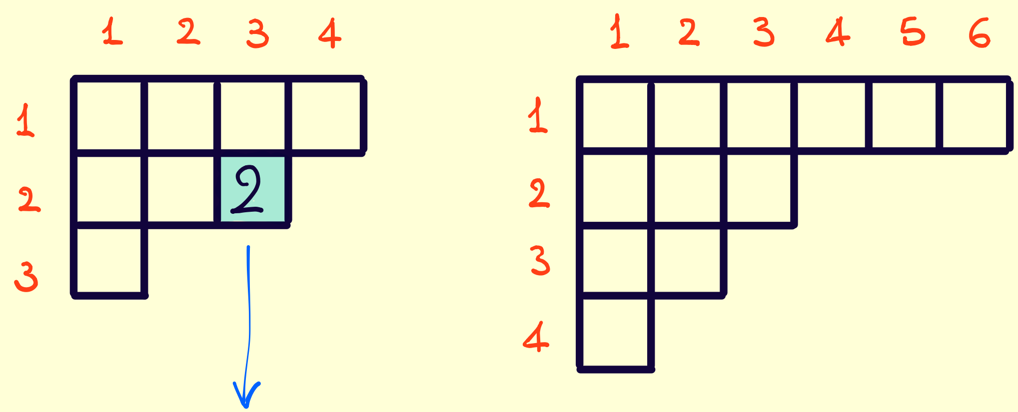
Example. $n = 20$, $r = 2$. $e = 3$. Bicharge $(\kappa_1, \kappa_2) = (1, 2) \in \mathbb{Z}_3^2$.

$\lambda_1 = ((4,3,1), (6,3,2,1))$

$\lambda_2 = ((7,2,1^2), (4,2^2,1))$

$\lambda_3 = ((6,2,1), (4^2,1^3))$

λ_1



$\#\text{column} - \#\text{row} + \kappa_1$

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$$\lambda_1 \qquad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 & \\ 3 & -1 & & & \end{array} \qquad \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 2 & 3 & & & \\ 3 & 0 & 1 & & & & \\ 4 & -1 & & & & & \end{array}$$

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Content of $\boldsymbol{\lambda}_1$: $\mathcal{C}(\boldsymbol{\lambda}_1) = (c_0(\boldsymbol{\lambda}_1), c_1(\boldsymbol{\lambda}_1), c_2(\boldsymbol{\lambda}_1))$

$= (6, 7, 7)$

Blocks of $\mathcal{H}_n$

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$\lambda_1 = ((4,3,1), (6,3,2,1)) \rightsquigarrow C(\lambda_1) = (6, 7, 7)$

$\lambda_2 = ((7,2,1^2), (4,2^2,1)) \rightsquigarrow C(\lambda_2) = (5, 8, 7)$

$\lambda_3 = ((6,2,1), (4^2,1^3)) \rightsquigarrow C(\lambda_3) = (6, 7, 7)$

Theorem. [Graham-Lehrer, Lyle-Mathas]

Let $\alpha, \beta \in \mathcal{P}(n,r)$. The Specht modules S^α, S^β belong to the same block of $\mathcal{H}_n$ if and only if $C(\alpha) = C(\beta)$.

B block of $\mathcal{H}_n$ $\rightsquigarrow$ $\mathcal{C}(B) = \left(c_0(B), c_1(B), \ldots, c_{e-1}(B) \right)$

↳ Content of B

Def. [Fayers]. We define the e-weight of B as the non-negative integer

$$w(B) = \left(\sum_{i=1}^{r} c_{k_i}(B) \right) - \frac{1}{2} \sum_{f \in \mathbb{Z}_e} \left(c_f(B) - c_{f+1}(B) \right)^2$$

§2. Basic Ariki-Koike Algebras

$r=1$: **Iwahori-Hecke algebra of type A** $\mathcal{H}_n^A$

- Specht modules : $\{ S^\lambda \mid \lambda \in \mathcal{P}(n) \}$

- Irreducible $\mathcal{H}_n^A$ - modules :

<u>Semisimple case</u>

$\{ S^\lambda \mid \lambda \in \mathcal{P}(n) \}$

<u>Non-semisimple case</u>

$\{ D^\lambda \mid \lambda \in \mathcal{P}(n)$ & λ is e-restricted $\}$

§2. Basic Ariki-Koike Algebras

$r = 1$: Iwahori-Hecke algebra of type A $\mathcal{H}_n^A$

$r = 2$: Iwahori-Hecke algebra of type B $\mathcal{H}_n^B$

Question :

Which blocks of $\mathcal{H}_n^A$ and $\mathcal{H}_n^B$ are multiplicity free (i.e. all decomposition numbers are zeros or ones) ?

$$w(B) = \left(\sum_{i=1}^{r} c_{\kappa_i}(B) \right) - \frac{1}{2} \sum_{f \in \mathbb{Z}_e} \left(c_f(B) - c_{f+1}(B) \right)^2$$

	$w=0$	$w=1$	$w=2$	$w=3$	$w \geq 4$
$\mathcal{H}_n^A$	✓	✓	✓ [Richards '96]	✓ [Fayers '08]	✗
$\mathcal{H}_n^B$	✓	✓ [Fayers '05]	✓ [Fayers '06]	✓ ? [Fayers, P. '23]	✗

Main tools of the proof.

Take B a 3-weight block of $\mathcal{H}_n^B$.

B $\qquad\qquad$ $[S^{\lambda} : D^{\mu}]$

Scopes equivalences

under certain hypotesis ☆
on λ and/or μ

prototypical block $\qquad B_0$ ✓ $\qquad [S^{\lambda_0} : D^{\mu_0}] \in \{0, 1\}$

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prototypical block B_0 ✓ $[S^{\lambda_0} : D^{\mu_0}] \in \{0,1\}$

Remaining cases :

$[S^\lambda : D^\mu]$, λ, μ not satisfying ☆

Cyclotomic Jantzen-Shaper Formula for Ariki-Koike algebras [James-Mathas]