Some representation theory of KMY algebras

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This is joint work with Nouf Alraddadi, my PhD student (graduated 2022). Some of this work appears in her PhD thesis.

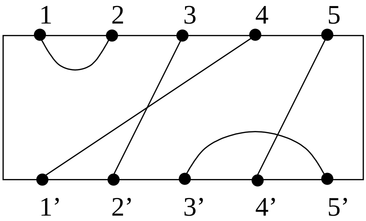
[KMY] = "On the geometrically defined extensions of the Temperley-Lieb category in the Brauer category", Kadar–Martin–Yu. Math Z. 2019.

A *Brauer diagram* is a picture/diagram of a pair partition of

$$\{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$$

E.g. $n = 5$: $\{\{1, 2\}, \{2', 3\}, \{1, 4\}, \{4', 5\}, \{3', 5'\}\}$

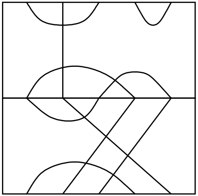
Can be viewed as a diagram, where all lines can be deformed isotopically.



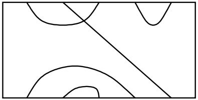
We fix n and use all possible diagrams as a basis for a vector space over a field (or ring) K.

To make an algebra we stack diagrams on top of each other to multiply:

E.g.

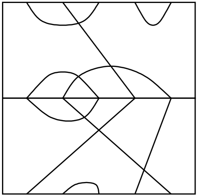


$=$

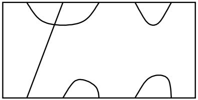


Brauer algebras and Brauer diagrams cont.

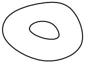
Might get loops:



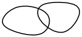
$= \delta$



The power of the parameter $\delta \in K$ records the number of loops removed.



$\rightsquigarrow \delta^2$



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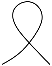
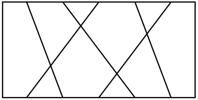


A twist gets untwisted — we don't record untangling of twists.

The Brauer algebra has the symmetric group algebra as a sub-algebra via permutation diagrams.

e.g. for n=5:



$$\leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

With this diagram realisation we have

$$K\mathfrak{S}_n \hookrightarrow \mathrm{Br}_n$$

where $\mathfrak{S}_n$ is the symmetric group on $\{1, 2, \ldots, n\}$ and Br_n is the Brauer algebra on n.

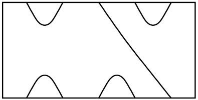
The representation theory of Br_n is at least as complicated as that for $K\mathfrak{S}_n$ and indeed it's unknown for char $K = p < n$.

There is another algebra inside Br_n, namely the *Temperley-Lieb algebra*, TL_n.

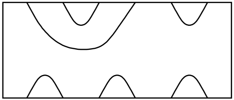
Take a *planar* representation of a diagram.

If this can be deformed in the plane to have *no* crossings then it's a *Temperley-Lieb diagram*.

e.g.

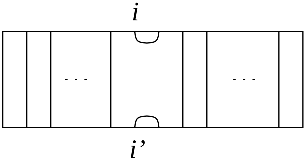


or



The representation theory of TL_n is *very* well understood. (Analogous to q-GL_2.)

The Brauer algebra is generated by the Temperley-Lieb generators, $1 \leq i \leq n-1$,

$u_i =$ 

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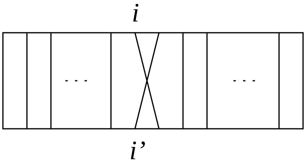
and the $K\mathfrak{S}_n$ generators $1 \leq i \leq n-1$,

$s_i =$ 

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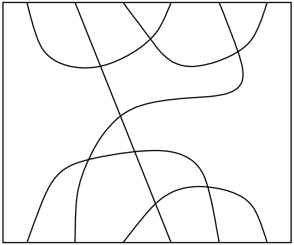
Are there algebras in between that might be easier to understand?

YES!!

Take a Brauer diagram. Define *height* as follows.

e.g. a diagram of height 2:



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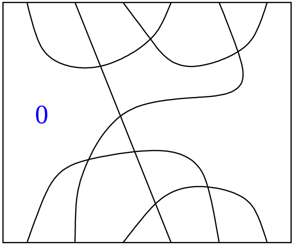

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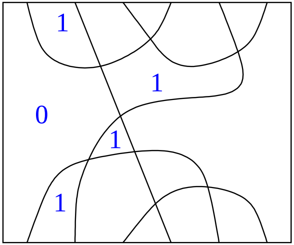

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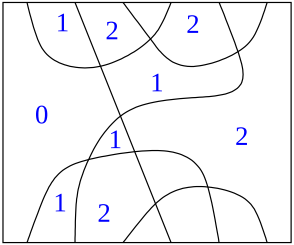

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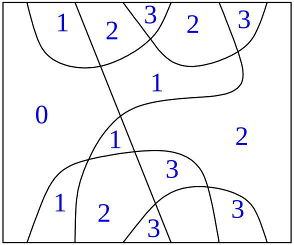

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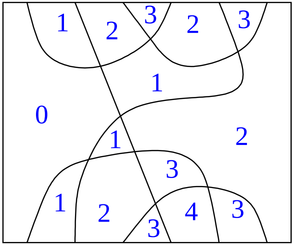

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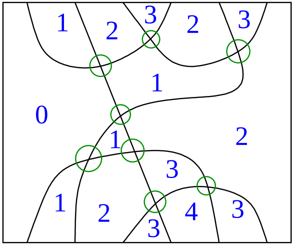

Brauer algebras and other algebras?

Are there algebras in between that might be easier to understand?

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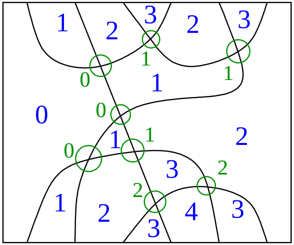
Take a Brauer diagram. Define *height* as follows.

e.g. a diagram of height 2:



Temperley-Lieb diagrams have height -1 (no crossings).

Let $h(d)$ be the height of a diagram d. KMY prove:

$$h(d_1 d_2) \leq \max\{h(d_1), h(d_2)\}$$

where d_1 and d_s are Brauer diagrams.

I.e. height can go *down* but never go *up* when we multiply.

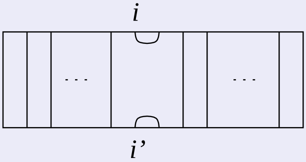
Thus diagrams of height at most l give a basis for a subalgebra, $J_{l,n}(\delta)$, of Br_n.

This algebra was first defined in [KMY] and I will henceforth call it the KMY-algebra.

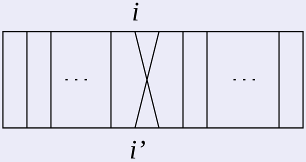
The KMY algebra has known generators

Theorem (Alraddadi)

$J_{l,n}(\delta)$ is generated by the Temperley-Lieb generators, $1 \leq i \leq n-1$,

$u_i =$ ,

and some of the $K\mathfrak{S}_n$ generators $1 \leq i \leq l+1$,

$s_i =$ ,

Note: s_{l+1} has height l.

We've already seen

$$J_{-1,n}(\delta) = \mathsf{TL}_n$$

$$J_{n-2,n}(\delta) = \mathsf{Br}_n$$

as Brauer diagrams have max height $n-2$.

Many of the properties of Br_n are also true for $J_{l,n}(\delta)$.

Let $\#(d)$ be the number of propagating lines in d.

Then: $\#(d_1 d_2) \leq \min\{\#(d_1), \#(d_2)\}$.

Let $I_m = \langle d \mid \#(d) \leq m \rangle \subseteq J_{l,n}(\delta)$.

These ideals filter the algebra.

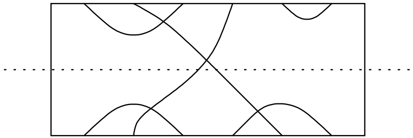
n even: $I_0 \leq I_2 \leq \cdots \leq I_{n-2} \leq I_n = J_{l,n}(\delta)$.

n odd: $I_1 \leq I_3 \leq \cdots \leq I_{n-2} \leq I_n = J_{l,n}(\delta)$.

$J_{l,n}(\delta)$ is *cellular* with a similar cell basis for Br_n. (Indeed, the algebra $J_{l,n}(\delta)$ is an iterated inflation of group algebras of symmetric groups.)

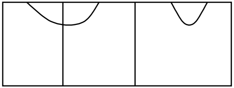
This uses half diagrams and bases for Specht modules for $K\mathfrak{S}_r$, $r \leq n$.

Half diagrams are formed from cutting the propagating lines.



Cell modules for the KMY algebra

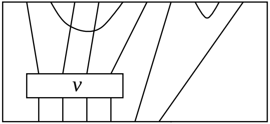
We ignore crossings in the propagating lines:



Let S^λ be a Specht module for $K\mathfrak{S}_m$, with $\lambda \vdash m$.

We form a basis for the cell module $\Delta_{l,n}(r,\lambda)$ by taking half diagrams with r propagating lines, $\lambda \vdash \min\{r, l+2\}$ and a basis for S^λ.

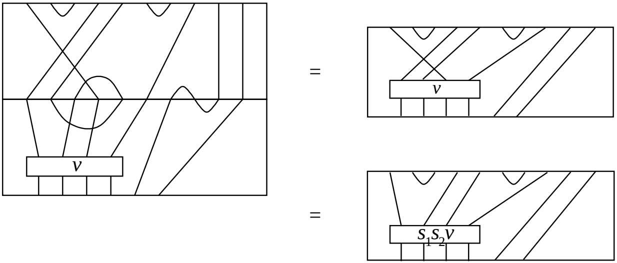
Let v be in a basis for S^λ. Elements look like:



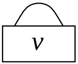
The half diagram has to have height at most l

The KMY algebra cell modules cont.

We get an action of $J_{l,n}(\delta)$ like so:



where $s_1 s_2 = (1,2)(2,3)$ is a permutation.

An arc acts as zero: 

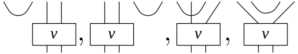
$= 0$. (Kills off propagating lines.)

The KMY algebra cell modules example

e.g. $n = 4$, $l = 0$ we get

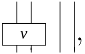
$$\Delta_{0,4}(0, \emptyset) = \Big\langle \quad \Big\rangle$$

$$\Delta_{0,4}(2, \lambda) = \Big\langle \quad \Big\rangle$$



$\lambda \vdash 2$ and $S^{\lambda} = \mathrm{Span}\{v\}$.

$$\Delta_{0,4}(4, \lambda) = \Big\langle \quad \Big\rangle$$



$\lambda \vdash 2$ and $S^{\lambda} = \mathrm{Span}\{v\}$.

We may use these bases to define Gram matrices and determinants.

If char $K > n$, the $J_{l,n}(\delta)$ satisfy the CMPX axioms and have nice localisation and globalisation functors.

This also shows that they are quasihereditary in this case.

KMY show that $J_{l,n}(\delta)$ is *generically* semisimple for $K = \mathbb{C}$.

Alraddadi shows that $J_{l,n}(\delta)$ is semisimple if $K = \mathbb{C}$, $l = 0$ and $\delta \notin \mathbb{R}$.

We have now removed the restriction on l to give:

Theorem (Alraddadi-Parker)

$J_{l,n}(\delta)$ is semisimple if $K = \mathbb{C}$ and $\delta \notin \mathbb{R}$.

Let

$$\Lambda_n^n = \{(n, \lambda) \mid \lambda \vdash l + 2\}$$
$$\Lambda_n^{n-2} = \{(n-2, \lambda) \mid \lambda \vdash \min\{n-2, l+2\}\}$$

We use the following:

Theorem (CMPX)

(Applied to $J_{l,n}(\delta)$.) Suppose that for all $n \geq 0$ and pairs of labels $(n, \lambda) \in \Lambda_n^n$ and $(n-2, \mu) \in \Lambda_n^{n-2}$ we have

$$\mathrm{Hom}(\Delta_n(n, \lambda), \Delta_n(n-2, \mu)) = 0.$$

Then each of the algebras $J_{l,n}(\delta)$ is semisimple.

How does this match up with known results for TL_n and Br_n?

For char $K = 0$ ($\mathfrak{S}_r$ always semisimple.)

- TL_n semisimple if $\delta = q + q^{-1}$, $q \in \mathbb{C}$ and q not a root of unity.
- Br_n is semisimple if $\delta \notin \mathbb{Z}$.

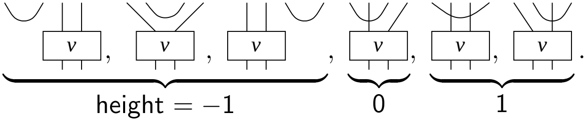
But $J_{l,n}(\delta)$ can be non-semisimple for $\delta \notin \mathbb{Z}$ and non-semisimple for δ not corresponding to singular Temperley-Lieb values.

e.g. in $n = 4$ case. We get zero Gram determinants if $\delta = \frac{-1 \pm \sqrt{17}}{2}$.

Example for $n = 4$

Take $n = 4$ and consider the cell module $\Delta_{I,4}(2,(2))$.

This module has basis where $S^{(2)} = \text{Span}\{v\}$.



And Gram matrix:

$$\left(\begin{array}{ccc|c|cc}
\delta & 1 & 0 & 1 & 1 & 1 \\
1 & \delta & 1 & 1 & 1 & 0 \\
0 & 1 & \delta & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & \delta & 0 & 1 \\
\hline
1 & 1 & 1 & 0 & \delta & 1 \\
1 & 0 & 1 & 1 & 1 & \delta
\end{array}\right)$$

Example for $n = 4$

The roots of the Gram determinant are given by

$$\delta = \begin{cases} 0, \pm\sqrt{2} & \text{if} \quad l = -1 \\ 0, 1, \frac{-1\pm\sqrt{17}}{2} & \text{if} \quad l = 0 \\ 0, 0, 0, -4, 2, 2 & \text{if} \quad l \geq 1 \end{cases}$$