Spherical Schubert Varieties

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PART 1

MUKASHIBANASHI

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Then R is an associative commutative graded ring w.r.t. the product

$$\chi^V \cdot \chi^W = \mathrm{ind}_{S_k \times S_{n-k}}^{S_n} \chi^V \times \chi^W$$

where χ^V is the character of the S_k-representation V. Likewise, χ^W is the character of the S_{n-k}-representation W.

$$\left(\begin{array}{lll} S_k & : & \text{the stabilizer of } \{1, \ldots, k\} \text{ in } S_n \\ S_{n-k} & : & \text{the stabilizer of } \{k+1, \ldots, n\} \text{ in } S_n \end{array} \right)$$

For $\lambda \vdash k$, $\mu \vdash n-k$, let

χ^λ : the character of the Specht module S^λ of S_k
χ^μ : the character of the Specht module S^μ of S_{n-k}

In R, we have

$$\chi^\lambda \cdot \chi^\mu = \bigoplus_{\tau \vdash n} c^\tau_{\lambda,\mu} \, \chi^\tau$$

for some nonnegative integers $c^\tau_{\lambda,\mu} \in \mathbb{N}$, called the Littlewood-Richardson numbers.

It is well-known that R is isomorphic to the algebra of symmetric functions in infinitely many commuting variables; $c^\tau_{\lambda,\mu}$'s can be computed via Schur's symmetric functions.

To really understand an algebra, one needs to understand its structure constants. In particular, we want to know when the Littlewood-Richardson numbers are as simple as 0 or 1. Of course, as a function of λ, μ, and τ, this question is already too general, so, we ask a more specific question:

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- For which $k \in \mathbb{N}$, do we have $c^{\tau}_{(k),(n-k)} \in \{0, 1\}$ for every $\tau \vdash n$? Equivalently, when is the induced trivial character $\mathrm{ind}^{S_n}_{S_k \times S_{n-k}} 1_{S_k \times S_{n-k}}$ multiplicity free?

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Definition

A subgroup $H \leqslant G$ is called a Gelfand subgroup if the induced trivial character $\mathrm{ind}^{G}_{H} 1_H$ is multiplicity free.

Remark

The induced character $\mathrm{ind}^{G}_{H} 1_H$ can be viewed as the character of the left G-module of $\mathbb{C}$-valued functions on G/H,

$$\mathbb{C}[G/H] = \{f : G/H \to \mathbb{C}\}.$$

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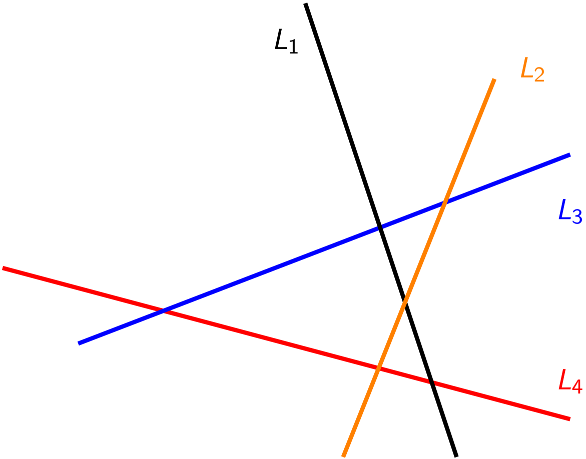

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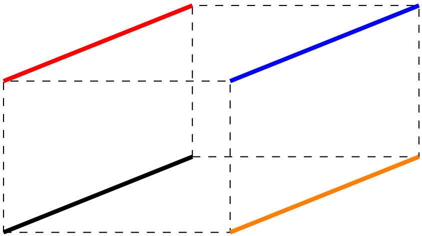

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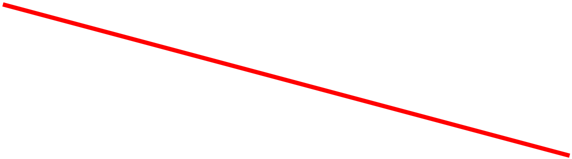

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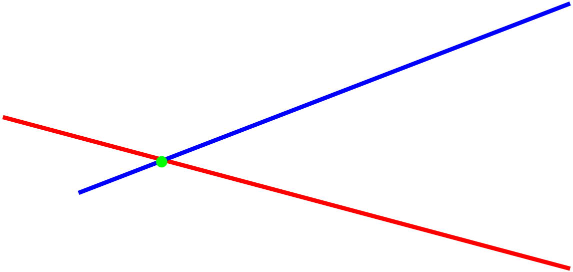

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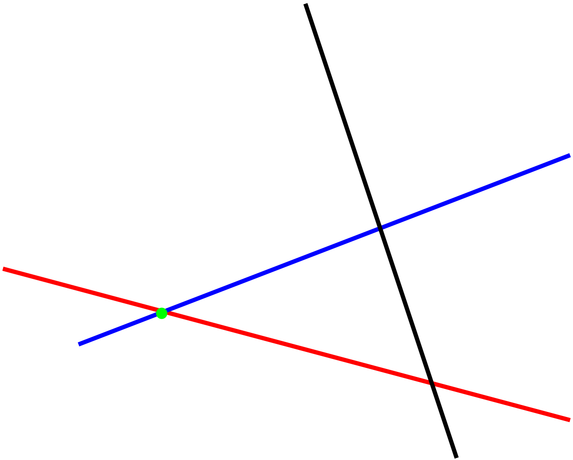

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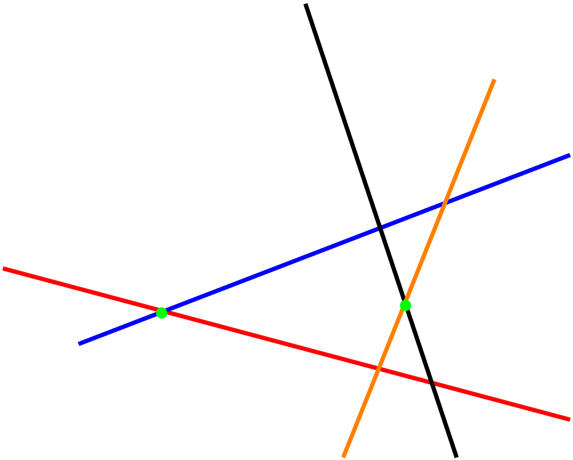

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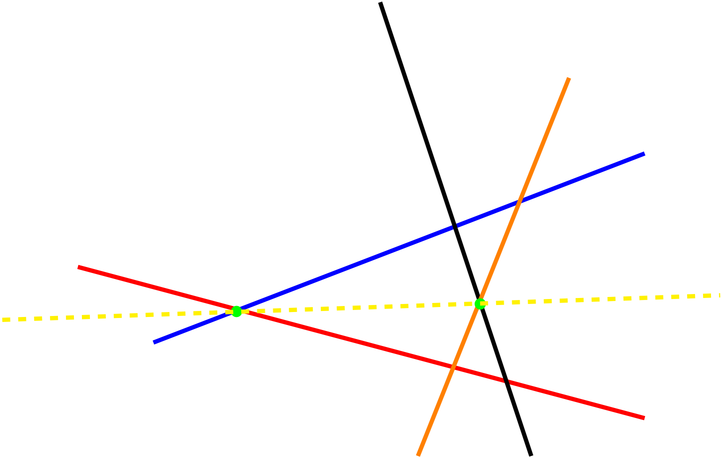

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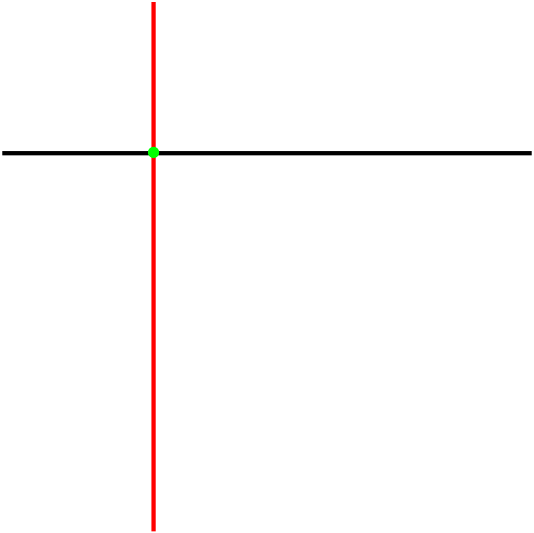

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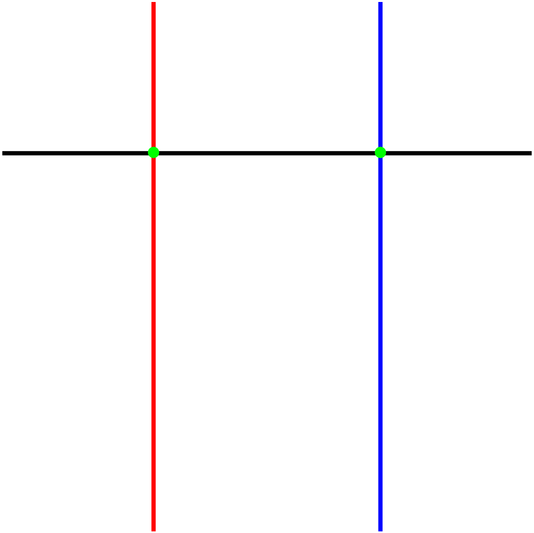

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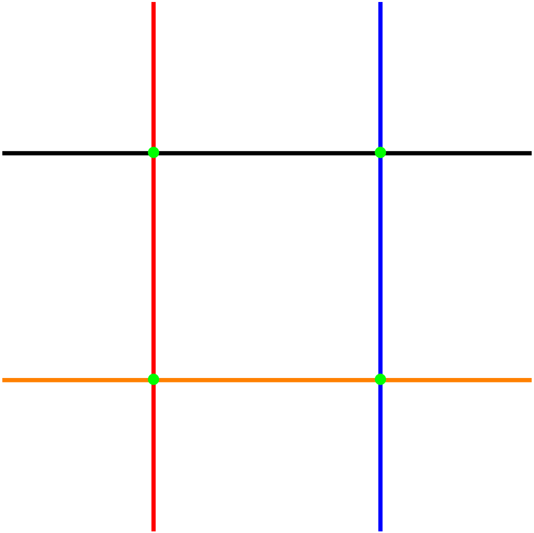

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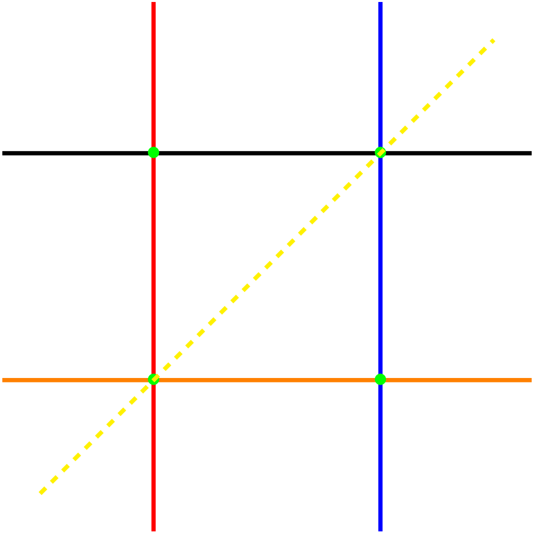

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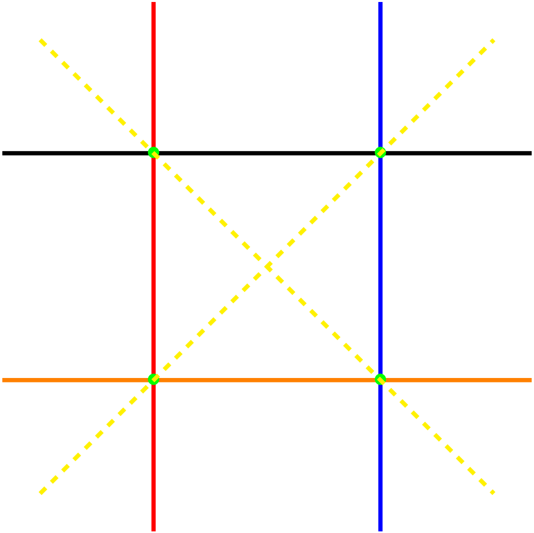

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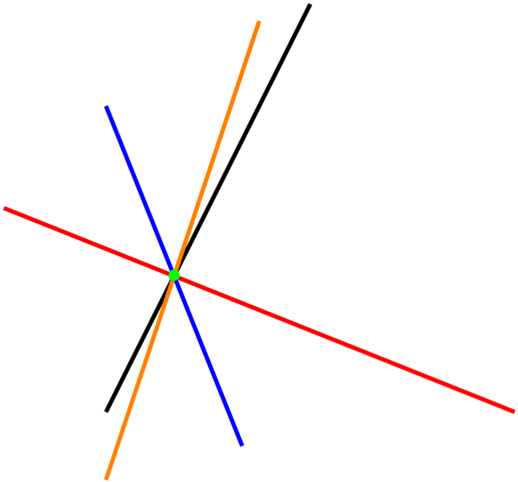

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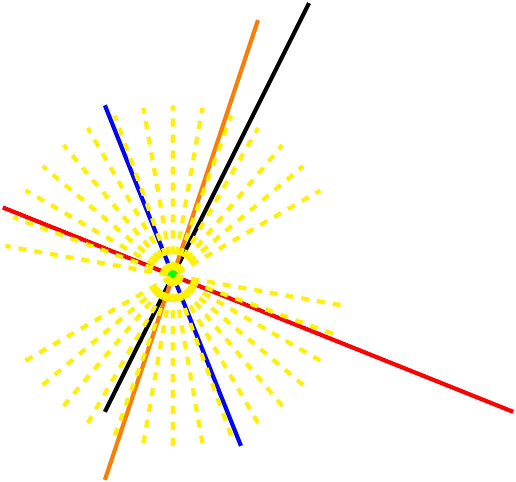

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No! If we consider ourselves living inside $\mathbb{P}^3$ instead of $\mathbb{C}^3$, all of that disorder disappears.. Here, $\mathbb{P}^3$ is the set of all 1-dimensional vector subspaces of $\mathbb{C}^4$.

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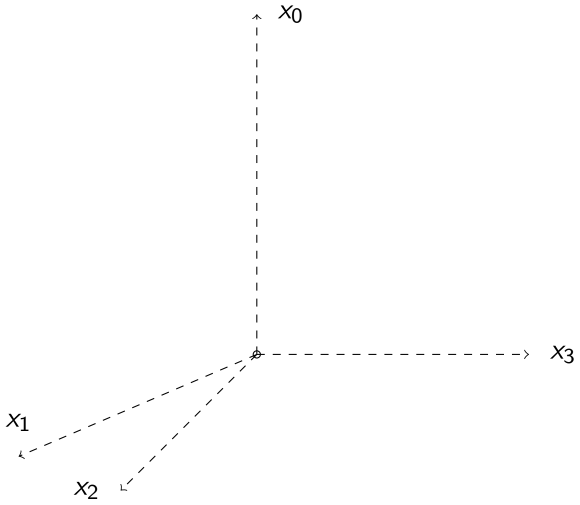

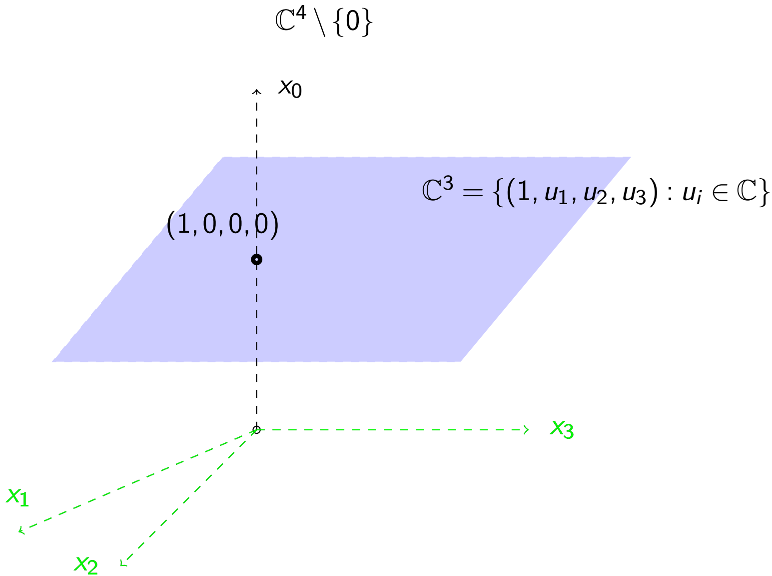
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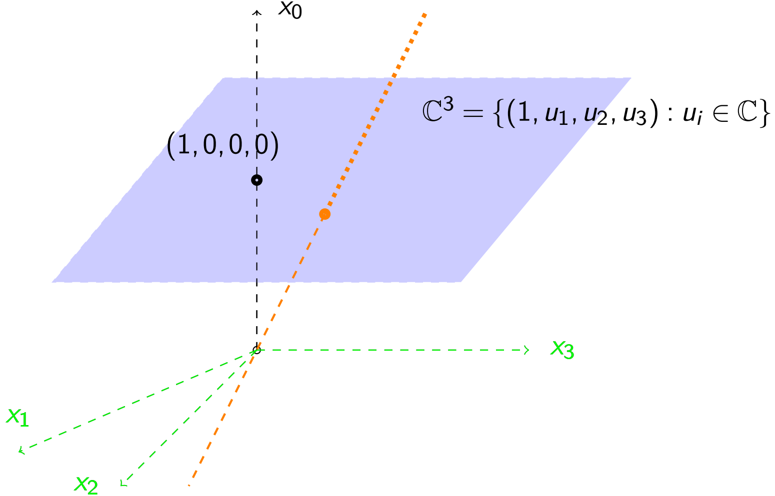

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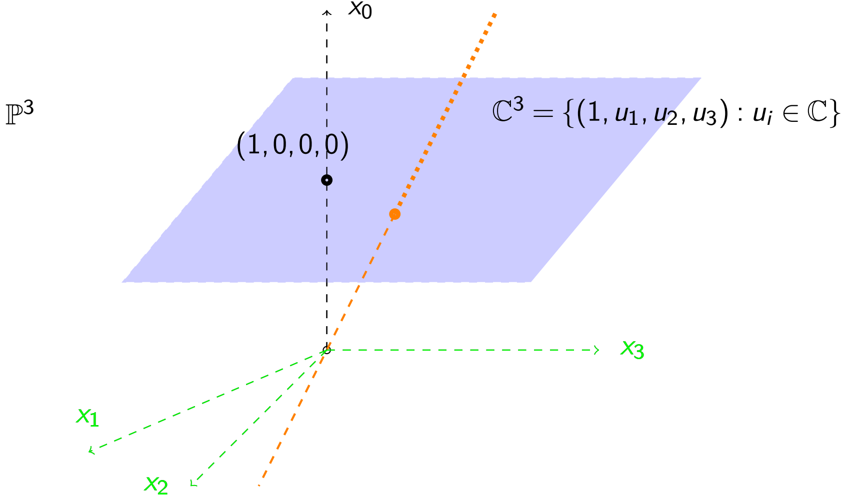

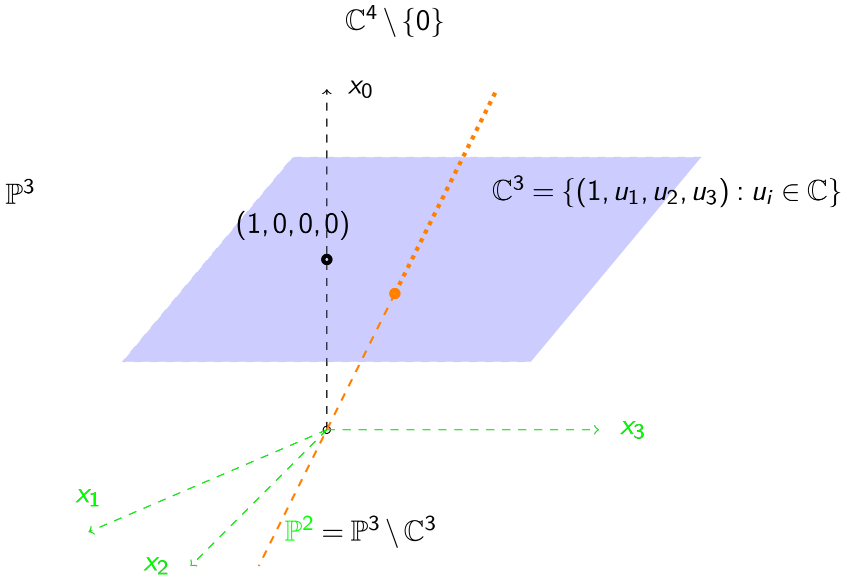
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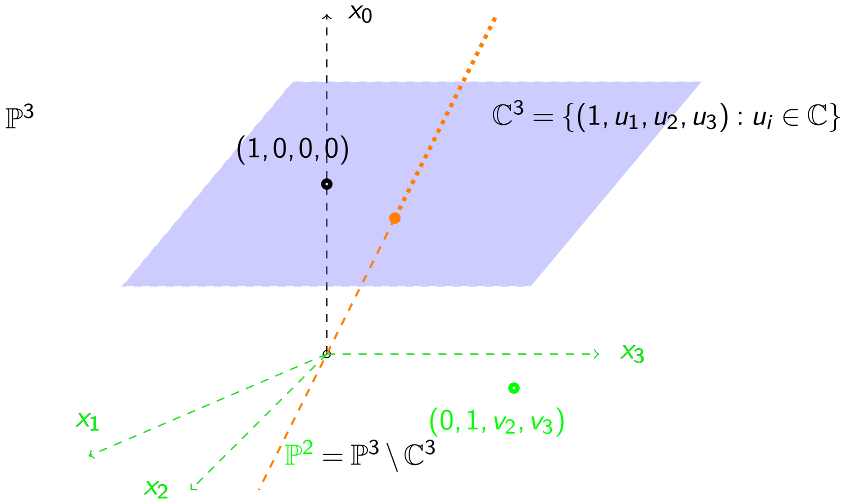

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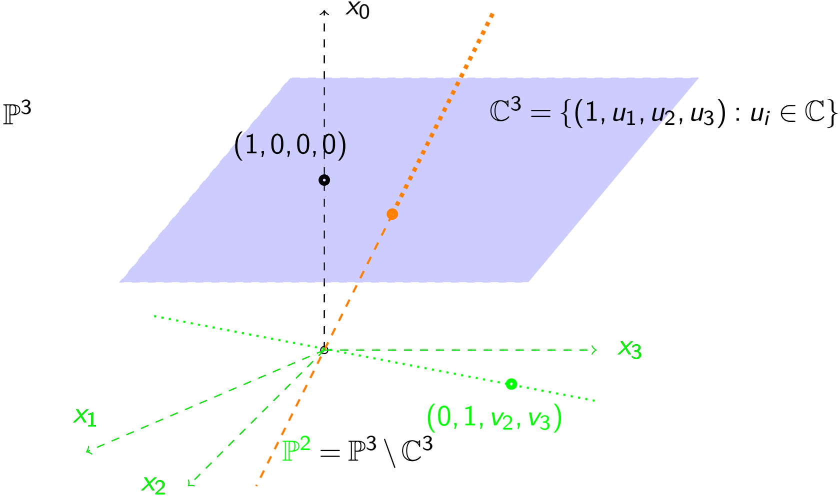

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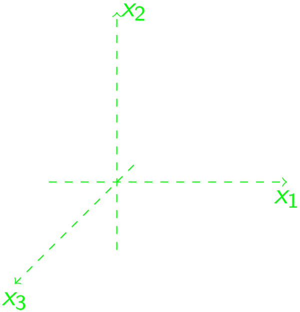

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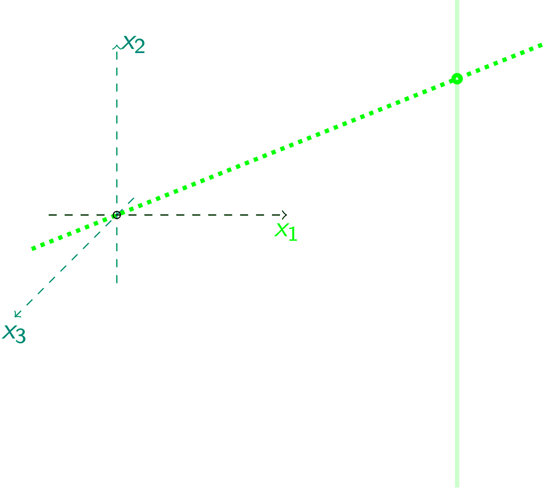

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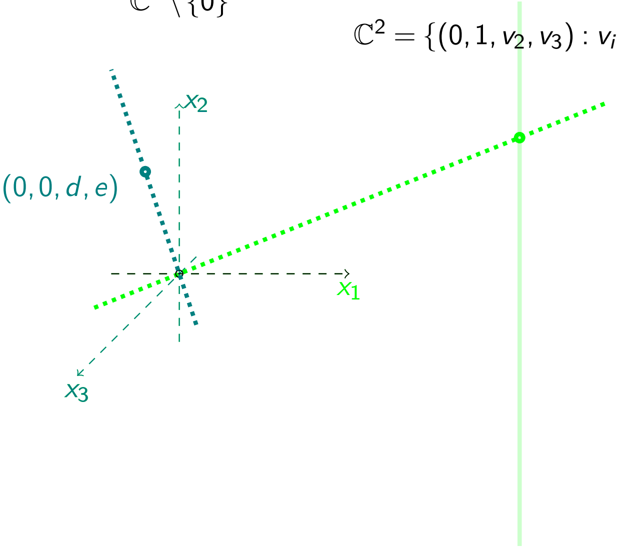

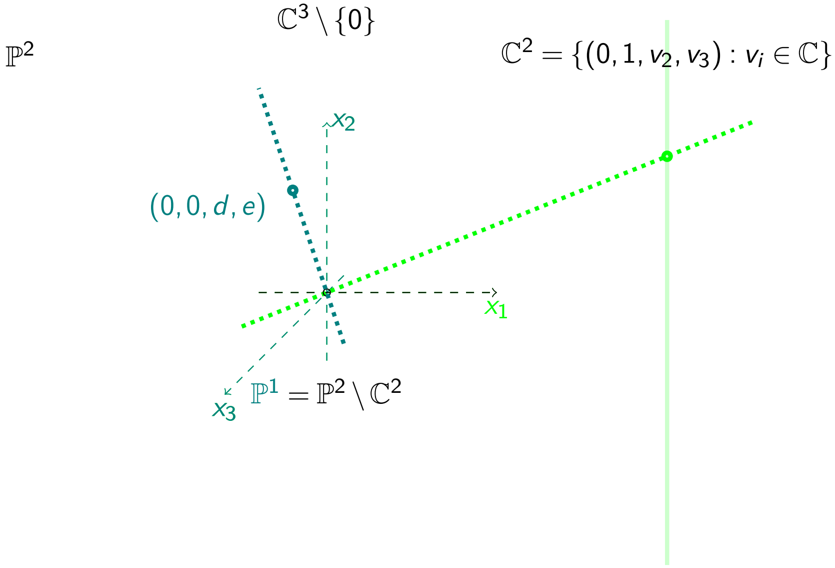
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Summary:

- $\mathbb{P}^n$ is the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$.

$$\begin{aligned}\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} &\longrightarrow \mathbb{P}^n \\ (u_0, \ldots, u_n) &\longmapsto \ell_{(u_0, \ldots, u_n)}\end{aligned}$$

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  Furthermore, this chain of subvarieties give a natural cellular decomposition,

$$\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^1 \sqcup \{pt\}.$$

An important generalization: There is a natural generalization of $\mathbb{P}^n$; it is called the Grassmann variety of d-dimensional subspaces of $\mathbb{C}^m$, denoted by $Gr(d, \mathbb{C}^m)$.

- $\mathbb{P}^n = Gr(1, \mathbb{C}^{n+1})$.
- $Gr(d, \mathbb{P}^m) := Gr(d+1, \mathbb{C}^{m+1})$.
- $Gr(d, \mathbb{C}^m)$ has a natural closed imbedding into a large dimensional projective space. For example, we know that

$$Gr(2, \mathbb{C}^4) \hookrightarrow \mathbb{P}^5.$$

  In other words, all Grassmann varieties are projective varieties.
- Similarly to $\{pt\} \hookrightarrow \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{P}^n$, $Gr(d, \mathbb{P}^m)$ has a natural poset of imbeddings of some subspecial varieties.

Let $A_\bullet = (A_0, \ldots, A_k)$ be a reference flag in $\mathbb{P}^n$,

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Note: If $B_\bullet : B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_k \subseteq \mathbb{P}^n$ is another reference flag such that $\dim B_i = \dim A_i$ for $i \in \{0, \ldots, k\}$, then there exists an automorphism of $Gr(k, \mathbb{P}^n)$ that maps $X(A_\bullet)$ isomorphically onto $X(B_\bullet)$.

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1. The image of p is a quadratic hypersurface in $\mathbb{P}^5$.
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We will apply this observation to our situation.

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- If H_i's are linearly independent, then $\cap_{i=1}^4 H_i$ is a line. In this case, we have $|\cap_{i=1}^4 H_i \cap Gr(1,\mathbb{P}^3)| = 2$ since in a projective space a line intersects a quadratic hypersurface at exactly two points, or at one point with multiplicity 2.

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- If H_i's are linearly dependent, then $\dim \cap_{i=1}^{4} H_i \geq 2$, hence, there are infinite number of points in $|\cap_{i=1}^{4} H_i \cap Gr(1, \mathbb{P}^3)|$.

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Theorem (The Schubert basis theorem.)

If Y denotes $Gr(d+1, \mathbb{C}^{n+1})$, then $H^{2p}(Y, \mathbb{Z})$ is a free abelian group. Furthermore, we have

1. *The duals of the homology classes corresponding to the Schubert varieties $X(a_0, \dots, a_d)$, where $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$, form a basis of $H^{2p}(Y, \mathbb{Z})$.*
2. *The odd dimensional cohomology groups of Y are trivial.*

Moreover, in the Schubert basis, the structure constants of the algebra $\bigoplus_{p=0}^{n} H^{2p}(Y, \mathbb{Z})$ are given by the Littlewood-Richardson numbers.

Let $d \in \{1, \ldots, n\}$. The d-th standard maximal parabolic subgroup of $\mathbf{GL}_{n+1}$ is

$$\mathbf{P}_d := \left\{ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} : \; A_{11} \in \mathbf{GL}_d, \; A_{22} \in \mathbf{GL}_{n+1-d} \right\} \subset \mathbf{GL}_{n+1}.$$

The standard Borel subgroup of $\mathbf{GL}_{n+1}$, denoted by B_{n+1}, is the subgroup that consists of upper triangular matrices in $\mathbf{GL}_{n+1}$.

Let $d \in \{1, \dots, n\}$. The d-th standard maximal parabolic subgroup of $\mathbf{GL}_{n+1}$ is

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Lemma

If $d \in \{1, \dots, n\}$, then there is a natural isomorphism

$$\phi : Gr(d, \mathbb{C}^{n+1}) \to \mathbf{GL}_{n+1}/\mathbf{P}_d.$$

Furthermore, the Schubert varieties in $Gr(d, \mathbb{C}^{n+1})$ correspond to the closures of the orbits of B_{n+1} in $\mathbf{GL}_{n+1}/\mathbf{P}_d$.

A (weak) composition of $n+1$ is a sequence of (nonnegative) positive integers $\mathbf{v} := (v_1, \ldots, v_r)$ such that $\sum_{i=1}^{r} v_i = n+1$. For example, $\mathbf{v} = (1, 3, 1)$ is a composition of 5.

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Associated with each composition of $n+1$ is a subgroup of the form

$$\mathbf{P_v} := \left\{ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{A_1 r} \\ 0 & A_{22} & \cdots & A_{A_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{rr} \end{bmatrix} : \prod_{i=1}^r A_{ii} \in \prod_{i=1}^r \mathbf{GL}_{v_i} \right\} \subset \mathbf{GL}_{n+1}.$$

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Any subgroup in $\mathbf{GL}_{n+1}$ that is conjugate to $\mathbf{P_v}$ will be called a parabolic subgorup.

Clearly, if $\mathbf{v} = (1, \ldots, 1)$, then $\mathbf{P}_{\mathbf{v}} = \mathbf{B}$. The parabolic subgroups containing $\mathbf{B}_{n+1}$ are called *standard with respect to* $\mathbf{B}_{n+1}$.

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Just as in the case of Grassmann varieties, the cohomology ring of a (partial) flag variety is completely determined by the Zariski closures of the orbits of $\mathbf{B}_{n+1}$. These subvarieties are also called the Schubert varieties.

Let λ be a character of $\mathbf{B}_{n+1}$. There is a natural action of $\mathbf{B}_{n+1}$ on the space $\mathbf{GL}_{n+1} \times \mathbb{C}$:

$$b \cdot (g, z) = (gb^{-1}, \lambda(b)z) \qquad (b \in \mathbf{B}_{n+1}, g \in \mathbf{GL}_{n+1},\ z \in \mathbb{C}).$$

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The quotient by this action defines a line bundle on the flag variety,

$$\mathbf{GL}_{n+1} \times^{\mathbf{B}_{n+1}} \mathbb{C} \longrightarrow \mathbf{GL}_{n+1}/\mathbf{B}_{n+1}. \tag{1}$$

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Theorem (Borel-Weil-Bott Theorem)

The space of global sections of (1) is isomorphic to the dual of the irreducible rational representation of $\mathbf{GL}_{n+1}$ *corresponding to* λ.

A related representation theoretic fact is the following: There is a natural projective embedding

$$\mathbf{GL}_{n+1}/\mathbf{B}_{n+1} \hookrightarrow \prod_{i=1}^{n} \mathbb{P}(\wedge^i \mathbb{C}^{n+1}).$$

Passing to the cone over the image of this closed embedding, we get the *total coordinate ring* of $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$.

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Passing to the cone over the image of this closed embedding, we get the *total coordinate ring* of $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$.

> Every irreducible rational representation of $\mathbf{GL}_{n+1} \times \mathbb{C}^*$ appears exactly once in the total coordinate ring of $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$.

Definition

Let G be a connected reductive group, and let V be rational G-module. V is said to be a multiplicity free G-module if every irreducible G-module appears at most once in V.

PART 2

SPHERICAL VARIETIES

Multiplicity-freeness and Borel subgroups.

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Theorem (Kimelfeld-Vinberg)

Let G be a connected reductive group, and let X be a quasi-affine homogeneous space of G. Then a Borel subgroup of G has an open orbit in X if and only if the coordinate ring $\mathbb{C}[X]$ is a multiplicity free G-module.

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Definition

An irreducible normal G-variety X is called a spherical G-variety if a Borel subgroup of G has an open orbit in X.

Spherical varieties.

The following theorem is proved by Brion in arbitrary characteristic, and around the same time by Vinberg in characteristic 0.

Theorem

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Theorem

A normal G-variety X is spherical iff the number of orbits of a Borel subgroup is finite.

Examples of spherical varieties include

- all partial flag varieties,
- normal toric varieties,
- normal reductive monoids,
- symmetric spaces as well as their equivariant embeddings.

What About Spherical Schubert Varieties?

The first question to answer here is "spherical with respect to which group action?"

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Recall that a Schubert variety is a $\mathbf{B}_{n+1}$-orbit closure in $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$. So, it is not stable under $\mathbf{GL}_{n+1}$. But we can consider the maximal reductive subgroups of the stabilizing subgroup.

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> Let X be a G-variety, and let $Y \subseteq X$ be a B-stable irreducible subvariety. Then $Stab_G(Y)$ is a parabolic subgroup of X.

Stabilizers of Schubert Varieties

Let S denote the set of simple transpositions

$$S = \{s_1 = (1,2), s_2 = (2,3), \ldots, s_n = (n, n+1)\}.$$

The elements of S generate S_{n+1}. For $I \subseteq S$, let W_I denote the subgroup of $W = S_{n+1}$ that is generated by the elements of I.

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In this notation, for every Schubert variety Y in $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$, there exists a subset $I = I(Y) \subset S$ such that the stabilizing parabolic subgroup of Y is of the form

$$\mathbf{P}_I := \mathbf{B}_{n+1} W_I \mathbf{B}_{n+1}.$$

The parabolic subgroup corresponding $I \subseteq S$.

Example

Let I be the subset $\{s_1, s_2, s_4, s_5, s_6\}$ in S_8. Then the corresponding parabolic subgroup is given by

$$\mathbf{P}_{(3,4,1)} := \left\{ \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \right\}$$

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Theorem (Bruhat-Chevalley decomposition)

Let W denote the symmetric group S_{n+1}, Then $\mathbf{GL}_{n+1} = \bigsqcup_{w \in W} \mathbf{B} w \mathbf{B}$. In particular, every Schubert variety Y in $\mathbf{GL}_{n+1}/\mathbf{B}_{n+1}$ is of the form $Y = \overline{\mathbf{B}_{n+1} w \mathbf{B}_{n+1}/\mathbf{B}_{n+1}}$ for some permutation $w \in S_{n+1}$.

Stabilizers of Schubert Varieties

If Y is the Schubert variety corresponding to the permutation $w \in S_{n+1}$ as in the theorem, then let us write $X_{w\mathbf{B}_{n+1}}$ instead of Y.

More generally, if Y is Schubert variety corresponding to the Weyl group element $w \in W$ in G/B, then we will write $Y = X_{wB}$.

The stabilizer of X_{wB} in G is found by the descent set of w^{-1}.

The strong order.

The Bruhat-Chevalley order on W (S_{n+1}) is defined by

$$v \leq w \iff X_{v\mathbf{B}_{n+1}} \subseteq X_{w\mathbf{B}_{n+1}}.$$

Note that $(W, \leq)$ is a graded poset.

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The interval between two elements $v, w \in W$ will be denoted by $[v, w]_{BC}$.

The (right) weak order.

Let v and w be two elements from W. The right weak order on W, denoted by $\leq_R$, is defined by the transitive closure of the following relation:

$$v \lessdot_R w \iff \exists\, s \in S \text{ such that } vs = w \text{ and } \ell(v) + 1 = \ell(w).$$

Remark

A permutation t which is conjugate to a simple transposition is called a reflection (or a transposition). The covering relations of the strong order are given by

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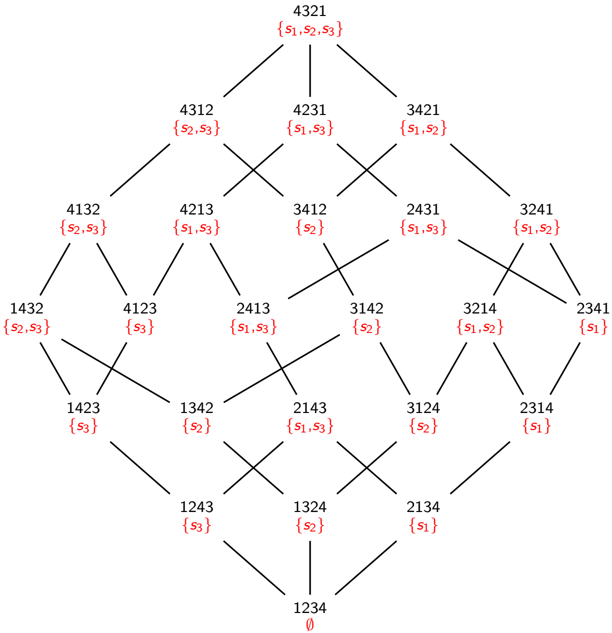
4321
{s_1,s_2,s_3}
4312
{s_2,s_3}
4231
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3421
{s_1,s_2}
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{s_2,s_3}
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2134
{s_1}
1234
∅

The standard Levi factor of $\mathbf{P}_I$.

The *standard Levi factor of* $\mathbf{P}_I$ is a maximal reductive subgroup in $\mathbf{P}_I$, denoted by $\mathbf{L}_I$ such that

$$\mathbf{P}_I = \mathbf{L}_I \ltimes \mathcal{R}_u(\mathbf{P}_I),$$

where $\mathcal{R}_u(\mathbf{P}_I)$ is the unipotent radical of $\mathbf{P}_I$. For example,

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$$\mathbf{L}_{(3,4,1)} := \left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \right\} \cong \mathbf{GL}_3 \times \mathbf{GL}_4 \times \mathbf{GL}_1.$$

Some recent results

Theorem

Let $w \in W$. If the lower intervals $[id, w]_R$ and $[id, w]_{BC}$ have the same underlying set of elements, then X_{wB} is a smooth spherical $\mathbf{L}_{I(w)}$-variety.

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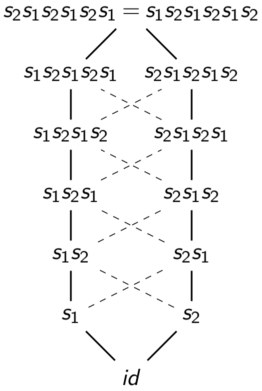
Theorem

Let G be one of the following algebraic groups: $\mathbf{GL}_2, \mathbf{GL}_3, \mathbf{GL}_4$, or $\mathbf{SO}_5$. Then every Schubert variety X_{wB} in G/B, where B is an appropriate Borel subgroup, is a spherical $L_{I(w)}$-variety.

In G_2.



This is the Bruhat-Chevalley order on the Weyl group of G_2. The right weak order is indicated by the solid lines.

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Theorem

Let X_{wB} be a Schubert variety in G/B, where G is of type G_2 Then, X_{wB} is a spherical $L_{I(w)}$-variety if and only if $w \in \{id, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_2s_1s_2s_2s_1s_2\}$.

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Corollary

Every smooth Schubert variety in a partial flag variety of a semisimple algebraic group of rank 2 is spherical.

A conjecture of Hodges and Yong.

Geometric proofs of the above theorems can be found in my paper (with Hodges)

https://arxiv.org/abs/1803.05515

This year, R. Hodges and A. Yong made some conjectures regarding the sphericality of Schubert varieties by introducing the "spherical elements" in Coxeter groups. Their article can be reached from here:

https://arxiv.org/abs/2007.09238

Further comments: Demazure characters.

I want to finish my presentation by showing a criteria for the sphericality of a Schubert variety; it is an immediate consequence of definitions and the works of Demazure and Lascoux-Schützenberger.

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G :	(complex) connected reductive group
B :	a Borel subgroup of G
T :	a maximal torus s.t. $T \subset B$
$X(T)^+$:	the monoid of dominant weights.

$X(T)^+$ parametrizes the irreducible rational representations of G. For $G = \mathbf{GL}_n$, we have

$$X(T)^+ \longleftrightarrow \text{integer partitions.}$$

If λ is a partition, Borel-Weil-Bott theorem gives us a line bundle on G/B, denoted by $\mathcal{L}(\lambda)$, whose space of global sections is an irreducible G-module.

Further comments: Demazure characters.

Recall that the total coordinate ring of G/B is the coordinate ring of the affine cone over G/B in $\prod_{i=1}^{n} \mathbb{P}(\wedge^i \mathbb{C}^{n+1})$, and that the total coordinate ring is the sum of all rational G-modules $H^0(G/B, \mathcal{L}(\lambda))$.

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The total coordinate ring of a Schubert subvariety $X_{wB} \subset G/B$ is a quotient of the total coordinate ring of G/B; it is the sum of all Demazure modules associated with w and λ, where λ runs over all partitions of n. In particular, a Schubert variety X_{wB} is a spherical variety if and only if for every partition λ of n the $\mathbf{L}_{I(w)}$-module $H^0(X_{wB}, \mathcal{L}(\lambda)|_{X_{wB}})$ is a multiplicity free.

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The character of $H^0(G/B, \mathcal{L}(\lambda))$ can be computed by the Weyl's character formula. Combinatorially, it is given by the Schur polynomial! In other words, the character ring of G is freely generated by the Schur polynomials.

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Around 1990, Lascoux and Schützenberger found combinatorial (polynomial) representatives for the Demazure characters; nowadays these polynomials are known as the key polynomials. We will denote then by $K_{w\cdot\lambda}$.

Further comments: Demazure characters.

Since a standard Levi subgroup of $\mathbf{GL}_n$ is a subgroup of the form $\mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_k}$, where $\sum_{i=1}^{k} n_i = n$, the character ring of $\prod_{i=1}^{k} \mathbf{GL}_{n_i}$ is freely generated by the products of Schur polynomials (in different variables). In particular, a key polynomial is a sum of products of certain Schur polynomials.

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Therefore, X_{wB} is a spherical variety if and only if one of the following equivalent conditions is satisfied:

- the total coordinate ring of X_{wB} is a multiplity free $\mathbf{L}_{I(w)}$-module;
- every key polynomial $K_{w\cdot\lambda}$, where λ is a partition of n, has a multiplicity free expansion in the product basis for the character ring of $\mathbf{L}_{I(w)}$.

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The sphericality problem for Schubert varieties is a combinatorial representation theory problem; by doing computer experimentations you can make significant progress on this and related problems.

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THE END

Hanashi wo kiite-kurete arigatou!

$$\begin{array}{lcl} Sym_n & : & \text{the invariant ring } \mathbb{C}[x_1,\ldots,x_n]^{S_n} \\ \mathscr{P}_n & : & \text{the set of all integer partitions of } n \\ \mathscr{C}_n & : & \text{the set of all weak-compositions of } n \end{array}$$

For $D = \{d_1 < \cdots < d_k\} \subseteq \{1,\ldots,n-1\}$, we set

$$\begin{array}{lcl} Sym_n(D) & : & Sym_{d_1} \otimes Sym_{d_2-d_1} \otimes \cdots \otimes Sym_{n-d_k} \\ \mathscr{P}_n(D) & : & \mathscr{P}_{d_1} \times \mathscr{P}_{d_2-d_1} \times \cdots \times \mathscr{P}_{n-d_k} \end{array}$$

and we set

$$\mathscr{C}_n(D) = \{\alpha \in \mathscr{C}_n : \ \mathrm{des}(\alpha) \subseteq D\},$$

where $\mathrm{des}(\alpha)$ is the set of indices $i \in \{1,\ldots,n-1\}$ such that $\alpha_i > \alpha_{i+1}$.

For $(\lambda^1, \dots, \lambda^k)$ from $\mathscr{P}_n(D)$, let $s_{(\lambda^1,\dots,\lambda^k)}$ denote the product

$$s_{\lambda^1}(x_1, \dots, x_{d_1}) \cdot \ldots \cdot s_{\lambda^k}(x_{n-d_k+1}, \dots, x_n),$$

where s_μ ($\mu \in \mathscr{P}_m$) is a Schur polynomial. Then

$$\{s_{(\lambda^1,\dots,\lambda^k)} : (\lambda^1, \dots, \lambda^k) \in \mathscr{P}_n(D)\} \tag{2}$$

gives a basis for $Sym_n(D)$.

Definition

An element $f \in Sym_n(D)$ is said to be D-multiplicity free if every coefficient of its expansion in the basis (2) is either 0 or 1.

Recall that a standard Levi subgroup of $\mathbf{GL}_n$ is a reductive subgroup of the form $\mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_k}$, where $\sum_{i=1}^{k} n_i = n$. The subset

$$I = \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{k-1}\} \subseteq \{1, \ldots, n-1\}$$

is uniquely determined by the composition $(n_1, \ldots, n_k)$ (and vice versa).

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Lemma

Let X_{wB} be a Schubert variety in G/B, where $G = \mathbf{GL}_n$. Let I denote $I(w)$, and let $\mathbf{L}_I$ denote the standard Levi factor of the stabilizer of X_{wB} in G. If λ is an integer partition, then the character of the $\mathbf{L}_I$-module $H^0(X_{wB}, \mathcal{L}(\lambda))$ is given by the key polynomial $K_{w\cdot\lambda}$.

Let w be as in the previous lemma. Then $K_{w \cdot \lambda}$ is an element of $Sym_n(D)$, where $D = \{1, \ldots, n-1\} \setminus I$, Let w be as in the previous lemma. Then $K_{w\cdot\lambda}$ is an element of $Sym_n(D)$, where $D = \{1, \ldots, n-1\} \setminus I$,

Corollary

The Schubert variety X_{wB} is $\mathbf{L}_I$-spherical if and only if for every partition $\lambda \vdash n$, the key polynomial $K_{w\cdot\lambda}$ is D-multiplicity free.

Key polynomials.

The i-th Demazure operator $D_i : \mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]$ is the operator defined by

$$D_i(f) = \frac{x_i \cdot f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - x_{i+1} f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}$$

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The key polynomial associated with $w \in S_n$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the polynomial defined by

$$K_{w \cdot \lambda} := D_{i_1} \cdots D_{i_r}(x_1^{\lambda_1} \cdots x_n^{\lambda_n}),$$

where the indices $i_1, \ldots, i_r$ are given by a (any) reduced expression of w,

$$w = s_{i_1} \cdots s_{i_r} \qquad (s_{i_j} = (i_j \, i_j + 1)).$$

Arigatōgozaimashita!