Non-commutative polynomials and Categorification

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Representation Theory of
Hecke Algebras and Categorification

Joint works with almost all of my coworkers

Recall the talk of Professor Kashiwara

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\widehat{\mathfrak{g}}$ the untwisted affine Kac-Moody algebra corresponding to $\mathfrak{g}$ (ex. $\mathfrak{g} = A_n \leftrightarrow \widehat{\mathfrak{g}} = A_n^{(1)}$).

The category $R^{\mathfrak{g}}\text{-gmod}$ of f.d (graded) modules over the quiver Hecke (KLR) algebra R has similar properties with the category $\mathscr{C}^{\widehat{\mathbf{g}}}$ of f.d integrable modules over the quantum affine algebra $\mathcal{U}_q'(\widehat{\mathbf{g}})$.

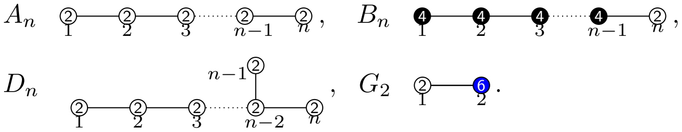
More precisely, those two categories are non-semisimple non-commutative monoidal categories and have R-matrices with $\mathbb{Z}$-invariants. However, $\mathscr{C}^{\widehat{\mathbf{g}}}$ has rigidity (module M in $\mathscr{C}^{\widehat{\mathbf{g}}}$ has the right dual $\mathscr{D}(M)$ and left dual $\mathscr{D}^{-1}(M)$), while $R^{\mathfrak{g}}\text{-gmod}$ has a natural $\mathbb{Z}[q^{\pm 1}]$-action.

In this talk, we mainly consider the side of quantum affine algebras. We denote by $\mathscr{C}_0^{\widehat{\mathbf{g}}}$ the "skeleton category" of $\mathscr{C}^{\widehat{\mathbf{g}}}$. Here the "skeleton category" means that every prime simple module in $\mathscr{C}^{\widehat{\mathbf{g}}}$ is a parameter shift of a prime simple module in $\mathscr{C}_0^{\widehat{\mathbf{g}}}$.

Notations

- For a statement P, $\delta(\mathsf{P}) = 1$ if P is true, 0 otherwise.
- I: the index set of the simple roots $\{\alpha_i \mid i \in I\}$ of $\mathfrak{g}$,
- $\triangle = (\triangle_0, \triangle_1)$: the Dynkin diagram of $\mathfrak{g}$,
- C : the Cartan matrix of $\mathfrak{g}$,
- Φ^+: the set of positive roots of $\mathfrak{g}$,
- W: the Weyl group of $\mathfrak{g}$, w_0: the longest element in W,
- $(\ ,\ )$: the symmetric bilinear form on the root lattice Q of $\mathfrak{g}$.

Example (Slightly different convention for $\triangle$)



Here $\textcircled{t}_k$ means that $(\alpha_k, \alpha_k) = t$ and note that there exist(s) non-trivial Dynkin diagram automorphism(s) $\vee$ $(\widetilde{\vee})$ on $\triangle$, when $\mathfrak{g}$ is simply-laced.

Dynkin quivers

A *Dynkin quiver* Q on $\triangle$ is a pair $(\triangle, \xi)$ consisting of $\triangle$ and a height function

$$\xi : \triangle_0 \to \mathbb{Z} \qquad (i \longmapsto \xi_i)$$

such that $|\xi_i - \xi_j| = 1$ when i and j are adjacent vertices in $\triangle$.

Example

$$Q : \overset{3}{\textcircled{2}} \longrightarrow \overset{2}{\textcircled{2}} \longrightarrow \overset{1}{\textcircled{2}} \text{ of } A_3, \quad Q : \overset{3}{\text{❹}} \longrightarrow \overset{2}{\text{❹}} \longrightarrow \overset{1}{\textcircled{2}} \text{ of } B_3.$$

More precisely, $\xi_1 = 3$, $\xi_2 = 2$, $\xi_3 = 1$ and $i \to j$ means $\xi_i = \xi_j + 1$.

$\mathfrak{g}_{\rm fin}$ for $\mathscr{C}_{\widehat{\mathfrak{g}}}$

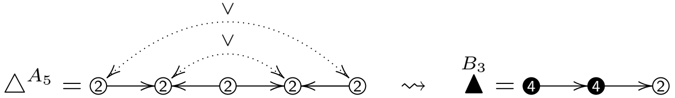
For each $\widehat{\mathfrak{g}}$, we assign the f.d simple Lie algebra $\mathfrak{g}_{\rm fin}$ of simply-laced type (unfolding of $\mathfrak{g}$) as follows:

$$\begin{array}{lclcl}
\widehat{\mathfrak{g}} = ADE_n^{(1)} & \leftrightarrow & \mathfrak{g}_{\rm fin} = ADE_n & \overset{\sigma = \mathrm{id}}{\rightsquigarrow} & \mathfrak{g} = ADE_n \\
\widehat{\mathfrak{g}} = B_n^{(1)} & \leftrightarrow & \mathfrak{g}_{\rm fin} = A_{2n-1} & \overset{\sigma = \vee}{\rightsquigarrow} & \mathfrak{g} = B_n, \\
\widehat{\mathfrak{g}} = C_n^{(1)} & \leftrightarrow & \mathfrak{g}_{\rm fin} = D_{n+1} & \overset{\sigma = \vee}{\rightsquigarrow} & \mathfrak{g} = C_n, \\
\widehat{\mathfrak{g}} = F_4^{(1)} & \leftrightarrow & \mathfrak{g}_{\rm fin} = E_6 & \overset{\sigma = \vee}{\rightsquigarrow} & \mathfrak{g} = F_4, \\
\widehat{\mathfrak{g}} = G_2^{(1)} & \leftrightarrow & \mathfrak{g}_{\rm fin} = D_4 & \overset{\sigma = \widetilde{\vee}}{\rightsquigarrow} & \mathfrak{g} = G_2.
\end{array}$$

Here $\sigma = \mathrm{id}$, $\vee$ or $\widetilde{\vee}$ is the Dynkin diagram automorphism on $\triangle^{\mathfrak{g}_{\rm fin}}$ yielding $\triangle^{\mathfrak{g}}$ via orbit.

Example

$B_3^{(1)} \quad \leftrightarrow$



Q-datum

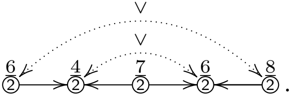
Definition (Fujita-O)

The Q-datum for $\widehat{\mathfrak{g}}$ is a triple $\mathcal{Q} = (\triangle^{\mathfrak{g}_{\mathrm{fin}}}, \sigma, \xi)$ such that (i) σ is the Dynkin diagram automorphism on $\triangle^{\mathfrak{g}_{\mathrm{fin}}}$ yielding $\triangle^{\mathfrak{g}}$ via σ and (ii) ξ is a function from $\triangle_0^{\mathfrak{g}_{\mathrm{fin}}} \to \mathbb{Z}$ satisfying certain conditions.

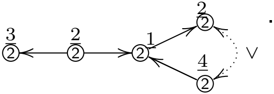
When $\sigma = \mathrm{id}$ and $\triangle$ is simply-laced, $\mathcal{Q}$ coincides with the notion of Dynkin quiver $Q := (\triangle, \xi)$. In this case, we understand a Dynkin quiver as a Q-datum.

Example

For $B_3^{(1)}$, $\vee$ is a non-trivial one on $\triangle^{A_5}$ and

$\mathcal{Q} =$ 

For $C_4^{(1)}$, $\vee$ is a non-trivial one on $\triangle^{D_5}$ and

$\mathcal{Q} =$ 

Heart subcategories

Theorem (...,Happel,Hernandez-Leclerc,..., Fujita-O)

For each Q*-datum* $\mathcal{Q} = (\triangle, \sigma, \xi)$ *of* $\widehat{\mathfrak{g}}$*, there exist* (i) *the (combinatorial) AR-quiver* $\Gamma^{\mathcal{Q}} = (\Gamma^{\mathcal{Q}}_0, \Gamma^{\mathcal{Q}}_1)$ *and* (ii) *injective* <u>*coordinate*</u> *maps* $\phi_{\mathcal{Q}}$ *and* $\widehat{\phi}_{\mathcal{Q}}$ *such that*

$$\begin{array}{ccccc} \Gamma^{\mathcal{Q}}_0 & \overset{1-1}{\longleftrightarrow} & \Phi^+_{\mathfrak{g}_{\mathrm{fin}}} & \overset{\phi_{\mathcal{Q}}}{\hookrightarrow} & I_{\mathfrak{g}_{\mathrm{fin}}} \times \mathbb{Z} \\ & & & \searrow & \uparrow \widehat{\phi}_{\mathcal{Q}} \\ & & & & \Phi^+_{\mathfrak{g}_{\mathrm{fin}}} \times \mathbb{Z} \end{array} \quad \textit{such that } \widehat{\phi}_{\mathcal{Q}}(-,0) = \phi_{\mathcal{Q}}.$$

Furthermore, the image $\widehat{\triangle}^{\sigma}_0$ *of* $\widehat{\phi}_{\mathcal{Q}}$ *parameterizes the fundamental reps in* $\mathscr{C}^{\widehat{\mathfrak{g}}}_0$*.*

Definition (Heart subcategories)

For each $\mathcal{Q}$ of $\widehat{\mathfrak{g}}$, define $\mathscr{C}^{\widehat{\mathfrak{g}}}_{\mathcal{Q}}$ the smallest full subcategory of $\mathscr{C}^{\widehat{\mathfrak{g}}}$ (i) stable by taking tensor products, subquotients and extensions, and (ii) containing the finite set of fundamental reps $\{L^{\mathcal{Q}}(\beta) \mid \beta \in \Phi^+_{\mathfrak{g}_{\mathrm{fin}}}\}$, where $L^{\mathcal{Q}}(\beta)$ is defined by the coordinate $\phi_{\mathcal{Q}}(\beta) = (i,p) \in I_{\mathfrak{g}_{\mathrm{fin}}} \times \mathbb{Z}$ of β via $\phi_{\mathcal{Q}}$.

Categorification

Theorem (Hernandez-Leclerc, Kang-Kashiwara-Kim, Kashiwara-O, Fujita, Scrimshaw-O)

Let $\mathcal{Q}$ be a Q-datum of $\widehat{\mathfrak{g}}$.

- (a) *Each fundamental rep in $\mathscr{C}_0^{\widehat{\mathfrak{g}}}$ is the k-th dual of $L^{\mathcal{Q}}(\beta)$ for unique $k \in \mathbb{Z}$ and $\beta \in \Phi^+_{\mathfrak{g}_{\mathrm{fin}}}$.*
- (b) *The Grothendieck ring $K(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}})$ of $\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}$ is isomorphic to $\mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g}_{\mathrm{fin}})^{\vee}|_{q=1} = \mathbb{C}[N]$.*
- (c) *There exists a SW-duality functor $\mathcal{F}_{\mathcal{Q}}$ from $R^{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod}$ to $\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}$ categorifying the isomorphism in (b) and sending simples to simples bijectively.*

Remark

For every $\widehat{\mathfrak{g}}$, the Grothendieck ring of each heart subcategory is of simply-laced type, even when $\widehat{\mathfrak{g}}$ is non simply-laced!

q-characters

For an untwisted affine $\widehat{\mathfrak{g}}$, let $\widehat{\triangle}_0^\sigma := \widehat{\phi}_{\mathcal{Q}}(\phi^+_{\mathfrak{g}_{\rm fin}} \times \mathbb{Z}) \subset I_{\mathfrak{g}_{\rm fin}} \times \mathbb{Z}$. and set

$$\mathcal{Y} := \mathbb{Z}[Y_{i,p}^{\pm 1} \mid (i,p) \in \widehat{\triangle}_0^\sigma].$$

Theorem ((q-character homomorphism): Frenkel-Reshetikhin)

a. *There exists an injective algebra homomorphism* $\chi_q : K(\mathscr{C}_0^{\widehat{\mathfrak{g}}}) \hookrightarrow \mathcal{Y}$.
b. $K(\mathscr{C}_0^{\widehat{\mathfrak{g}}}) \simeq \mathbb{Z}\big[[L(Y_{i,p})] \mid (i,p) \in \widehat{\triangle}_0^\sigma\big]$. *Here* $L(Y_{i,p})$ *denotes the fundamental rep. in* $\mathscr{C}_0^{\widehat{\mathfrak{g}}}$, *labelled by* $(i,p) \in \widehat{\triangle}_0$.

Theorem (Chari-Pressley, Kashiwara, Varagnolo-Vasserot)

Every simple module in $\mathscr{C}_0^{\widehat{\mathfrak{g}}}$ *appears as a head of certain ordered product fundamental reps. Hence every simple module in* $\mathscr{C}_0^{\widehat{\mathfrak{g}}}$ *is labeled by a unique (dominant) monomials in* $Y_{i,p}$*'s uniquely. Thus we can write a simple module as* $L(m)$ *for some (dominant) monomials* $m = \prod Y_{i,p}^{m_{i,p}}$ $(m_{i,p} \in \mathbb{Z}_{\geq 0})$.

Non-commutative t-quantization

For $\widehat{\mathfrak{g}} = ADE_n^{(1)}$, Nakajima and Varagnolo-Vasserot constructed t-deformation $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ of $\chi_q(K(\mathscr{C}_0^{\widehat{\mathfrak{g}}})) \simeq K(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ from a geometric point of view, which is a non-commutative $\mathbb{Z}[t^{1/2}]$-algebra. Then Hernandez constructed $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ for all $\widehat{\mathfrak{g}}$ uniformly in a purely algebraic way.

$$\mathcal{Y}_t \hookleftarrow \mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}}) \xrightarrow{t=1} \chi_q(K(\mathscr{C}_0^{\widehat{\mathfrak{g}}})) \hookrightarrow \mathcal{Y}$$

where $\mathcal{Y}_t$ is a non-commutative Laurent polynomial ring over $\mathbb{Z}[t^{\pm 1/2}]$, also called quantum torus, generated by $\{\widetilde{Y}_{i,p}^{\pm} \mid (i,p) \in \widehat{\triangle}_0^{\sigma}\}$ subject to following relations:

$$\widetilde{Y}_{i,p}\widetilde{Y}_{i,p}^{-1} = \widetilde{Y}_{i,p}^{-1}\widetilde{Y}_{i,p} = 1, \quad \widetilde{Y}_{i,p}\widetilde{Y}_{j,s} = t^{\mathcal{N}(i,p;j,s)}\widetilde{Y}_{j,s}\widetilde{Y}_{i,p}.$$

Here $\mathcal{N}(\ ;\ )$ is an anti-symmetric form on $\widehat{\triangle}_0 \times \widehat{\triangle}_0$ determined by the inverse of the quantum Cartan matrix $\mathsf{C}(q)$ of C:

$$\mathsf{C}(q) = (\mathsf{c}_{ij}(q)) \quad \text{where } \mathsf{c}_{ij}(q) = \delta_{i,j}(q^{d_i} + q^{-d_i}) + (1-\delta_{i,j})[\mathsf{c}_{i,j}]_q$$

Bases of Quantum Grothendieck ring

We call $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ the quantum Grothendieck ring of $\widehat{\mathfrak{g}}$.

Theorem (Nakajima, Hernandez)

(a) *For each fundamental rep* $L(Y_{i,p}) \in K(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$*, there exists* $L_t(Y_{i,p}) \in \mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ *such that* $L_t(Y_{i,p})|_{t=1} = \chi_q(L(Y_{i,p}))$.
(b) $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ *has an basis*

$$\mathbf{E}_t := \{E_t(m) := \prod L_t(Y_{i,p})^{m_{i,p}} \mid m = \prod Y_{i,p}^{m_{i,p}},\ m_{i,p} \in \mathbb{Z}_{\geq 0}\},$$

which is called the basis of (q,t)*-characters of standard modules.*
(c) *There exists a unique "canonical"* $\mathbb{Z}[q^{\pm 1/2}]$*-basis*

$$\mathbf{L}_t = \{L_t(m) \mid \overline{L_t(m)} = L_t(m), m \text{ is a dominant monomial}\}$$

such that

$$E_t(m) = L_t(m) + \sum_{m' <_N m} P_{m,m'}(t) L_t(m) \text{ for some } P_{m,m'}(t) \in t\mathbb{Z}[t].$$

$\mathbf{L}_t$ *is called the canonical basis of simple* (q,t)*-characters* $L_t(m)$*'s.*

Positivity conjectures and corresponding results

Conjecture (Analog of Kazhdan-Lusztig and Positivities)

- (a) *$L_t(m)$ recovers $\chi_q(L(m))$. That is, $L_t(m)|_{t=1} = \chi_q(L(m))$.*
- (b) *Coefficients of $L_t(m)$ are contained in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.*
- (c) *$P_{m,m'}(t)$ is contained in $t\mathbb{Z}_{\geq 0}[t]$.*
- (d) *The structure constants of $\mathbf{L}_t$ are contained in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.*

Theorem (Nakajima, Varagnolo-Vasserot)

For $\widehat{\mathfrak{g}} = ADE_n^{(1)}$, we have

- (i) *The first, second and third conjectures hold (Nakajima).*
- (ii) *The fourth conjecture holds (Varagnolo-Vasserot).*

Conjectures for heart subcategories (including $BCFG^{(1)}$)

Theorem (Hernandez-Leclerc,Hernandez-Oya,Fujita-Hernandez-O-Oya,+)

For every $\widehat{\mathfrak{g}}$ and a Q-datum $\mathcal{Q}$ of $\widehat{\mathfrak{g}}$, we have a $\mathbb{Z}$-algebra isomorphism

$$\Psi_{\mathcal{Q}} : \mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g}_{\mathrm{fin}})^{\vee} \xrightarrow{\sim} \mathcal{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}) \qquad (\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}])$$

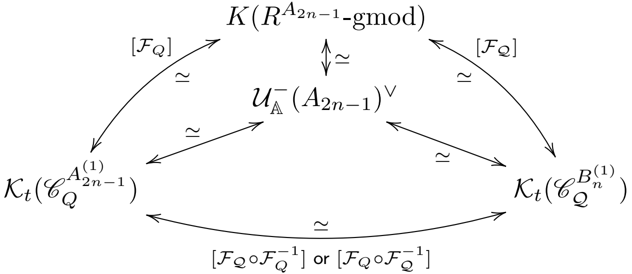
sending

1. $q^{\pm 1/2}$ *to* $t^{\pm 1/2}$,
2. *the dual-canonical/upper-global basis of* $\mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g}_{\mathrm{fin}})^{\vee}$ *to* $\mathbf{L}_{t,\mathcal{Q}} := \mathbf{L}_t \cap \mathcal{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}})$,
3. *the dual PBW-basis associated with reduced expressions of* $w_0 \in \mathsf{W}_{\mathfrak{g}_{\mathrm{fin}}}$ *adapted to $\mathcal{Q}$ to* $\mathbf{E}_{t,\mathcal{Q}} := \mathbf{E}_t \cap \mathcal{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}})$ *(up to* $\mathbb{Z}[t^{\pm 1/2}]^{\times}$*),*

Hence the conjectures hold for $\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}$ *including* $BCFG^{(1)}$ *with* $\mathcal{F}_{\mathcal{Q}}$.

Ending remark of the first half

As emphasized, the Grothendieck rings of heart subcategories are of simply-laced type, even in the non simply-laced affine type $\widehat{\mathfrak{g}}$. For instance,



Question

1. *The conjectures for whole $BCFG^{(1)}$?*
2. *Non-commutative polynomial rings including $\mathcal{U}_{\mathbb{A}}^{-}(BCFG_n)^{\vee}$?*

Note that quantum torus and positivity conjectures also appear in the (quantum) cluster algebra theory.

Presentation of $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$

Note that $\mathbb{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}) := \mathbb{Q}(t^{1/2}) \otimes_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}}) \simeq \mathcal{U}_q^-(\mathfrak{g}_{\mathrm{fin}})$ and dual PBW-vectors adapted to $\mathcal{Q}$ are categorified by fundamental reps $L^{\mathcal{Q}}(\beta)$ $(\beta \in \Phi^+_{\mathfrak{g}_{\mathrm{fin}}})$. Thus the ring $\mathbb{K}_t(\mathscr{C}_{\mathcal{Q}}^{\widehat{\mathfrak{g}}})$ is generated by $L_t^{\mathcal{Q}}(\alpha_i)$ $(i \in I)$ subject to the quantum Serre's relation.

Theorem (Hernandez-Leclerc ($ADE^{(1)}$), Fujita-Hernandez-O-Oya ($BCFG^{(1)}$))

For each $\widehat{\mathfrak{g}}$, $\mathbb{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}}) := \mathbb{Q}(t^{1/2}) \otimes_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ is generated by

$$\{f_{i,p} \leftrightarrow \mathfrak{D}_t^p(L_t^{\mathcal{Q}}(\alpha_i)) \mid \forall (i,p) \in I_{\mathfrak{g}_{\mathrm{fin}}} \times \mathbb{Z}\}$$

subject to following relations: (Choice of $\mathcal{Q}$ does not matter)

$$\sum_{s=0}^{1-\mathbf{c}_{i,j}} (-1)^s \begin{bmatrix} 1-\mathbf{c}_{i,j} \\ s \end{bmatrix}_t f_{i,k}^{1-\mathbf{c}_{i,j}-s} f_{j,k} f_{i,k}^s = 0,$$

$$f_{i,k} f_{j,k+1} = t^{-(\alpha_i,\alpha_j)} f_{j,k+1} f_{i,k} + (1 - t^{-(\alpha_i,\alpha_i)})\delta_{i,j},$$

$$f_{i,k} f_{j,l} = t^{(-1)^{k+l}(\alpha_i,\alpha_j)} f_{j,l} f_{i,k},$$

for $\mathbf{C} = (\mathbf{c}_{i,j})$ of type $\mathfrak{g}_{\mathrm{fin}}$, $i,j \in I$ and $k,l \in \mathbb{Z}$ with $l > k+1$.

Consequences

That means,

- (a) even though $\widehat{\mathfrak{g}}^{(1)}$ and $\widehat{\mathfrak{g}}^{(2)}$ are different, $\mathbb{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}^{(i)}})$ are isomorphic if $\mathfrak{g}_{\mathrm{fin}}^{(1)} = \mathfrak{g}_{\mathrm{fin}}^{(2)}$.
- (b) even though $\widehat{\mathfrak{g}}$ is non-simply-laced, $\mathbb{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ is simply-laced.

To show the quantum Boson-relations, we have used the Λ-theory for quantum affine algebras developed by Kashiwara-Kim-O-Park.

Theorem (Fujita-Hernandez-O-Oya)

For $\widehat{\mathfrak{g}}^{(1)}, \widehat{\mathfrak{g}}^{(2)}$ with $\mathfrak{g}_{\mathrm{fin}}^{(1)} = \mathfrak{g}_{\mathrm{fin}}^{(2)}$, there exist $\mathbb{Z}[t^{\pm 1/2}]$-algebra isomorphisms

$$\Xi_{\mathcal{Q}^{(1)},\mathcal{Q}^{(2)}} : \mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}^{(1)}}) \to \mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}^{(2)}})$$

sending $\mathbf{L}_t^{(1)}$ to $\mathbf{L}_t^{(2)}$ for any Q-data $\mathcal{Q}^{(i)}$ of $\widehat{\mathfrak{g}}^{(i)}$. Thus the third and fourth conjectures hold for $BCFG^{(1)}$ also.

In particular, when $\widehat{\mathfrak{g}} = B_n^{(1)}$...

Theorem (Kashiwara-Kim-O)

There exists an exact monoidal functor

$$\Phi_{A,B} : \mathscr{C}_0^{A_{2n-1}^{(1)}} \to \mathscr{C}_0^{B_n^{(1)}}$$

sending simple modules to simple modules bijectively.

Theorem (Hernandez-Oya,Fujita-Hernandez-O-Oya)

There exist Q*-data* Q *of* $A_{2n-1}^{(1)}$ *and* $\mathcal{Q}$ *of* $B_n^{(1)}$ *such that*

$$[\Phi_{A,B}] = \Xi_{Q,\mathcal{Q}}|_{t=1}.$$

Thus the first and second conjectures also hold for $B_n^{(1)}$.

This technique is referred to as the propagation of positivities.

Monoidal categorification of cluster algebra

Hernandez-Leclerc proved that the Grothendieck ring $K(\mathscr{C}^{\widehat{\mathfrak{g}}}_{\le 0})$ of "negative-half subcategory" $\mathscr{C}^{\widehat{\mathfrak{g}}}_{\le 0}$ of $\mathscr{C}^{\widehat{\mathfrak{g}}}_{0}$ has a cluster algebra structure.

Theorem (Kashiwara-Kim-O-Park)

- (a) $K(\mathscr{C}^{\widehat{\mathfrak{g}}}_{0})$ *has a cluster algebra structure* $\mathscr{A}$ *of skew-symmetric type.*
- (b) *Each cluster monomial of* $K(\mathscr{C}^{\widehat{\mathfrak{g}}}_{0})$ *is categorified by a* <u>*real*</u> *simple module in* $\mathscr{C}^{\widehat{\mathfrak{g}}}_{0}$*; that is, the category* $\mathscr{C}^{\widehat{\mathfrak{g}}}_{0}$ *provides a monoidal categorification of* $\mathscr{A}$*.*
- (c) *Various subcategories* $\mathcal{C}$*, including* $\mathcal{C} = \mathscr{C}^{\widehat{\mathfrak{g}}}_{\le 0}$, $\mathscr{C}^{\widehat{\mathfrak{g}}}_{\le \xi}$ *and* $\mathscr{C}^{\widehat{\mathfrak{g}}}_{Q}$*, play the same roles.*

Remark

Once it is proved that a category $\mathcal{C}$ *provides a monoidal categorification of a cluster algebra* $\mathscr{A}$*, then the Laurent positivity and the Laurent independency hold for* $\mathscr{A}$*.*

Almost all of remaining conjectures (for $CFG^{(1)}$)

Theorem (Fujita-Hernandez-O-Oya)

- (a) *$\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$ has the quantum cluster algebra structure $\mathscr{A}_t$ of skew-symmetric type.*
- (b) *Each isomorphism $\Xi_{\mathcal{Q}^{(1)},\mathcal{Q}^{(2)}}$ can be interpreted as a sequence of mutations $+$ permutations.*

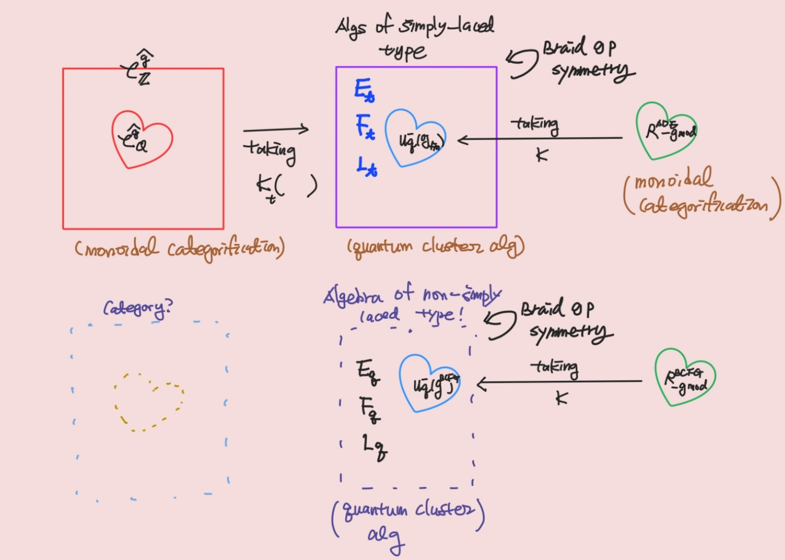
Thus we can conclude the followings:

1. *The second conjecture holds.*
2. *The first conjecture holds for reachable modules $L(m)$. Here $L(m)$ is reachable if $[L(m)]$ appears as a cluster monomial of $\mathscr{A}$.*

Remark

Still, every $\mathcal{K}_t$ (or K) we dealt with is of simply-laced (or skew-symmetric).

Gap


Algs of simply-laced type
Braid gp symmetry
taking
taking
K
(monoidal categorification)
(quantum cluster alg)
Category?
Algebra of non-simply laced type!
Braid gp symmetry
taking
K
(quantum cluster alg)

Toward the second question (very briefly)

From the co-works with Kashiwara and Jang-Lee, we have constructed new quantum torus $\mathcal{X}_q$ over $\mathbb{Z}[q^{\pm 1/2}]$ and its subalgebra $\mathfrak{K}_q(\mathfrak{g})$ for every $\mathfrak{g}$. In particular, it is isomorphic to $\mathcal{K}_t(\mathscr{C}_0^{\widehat{\mathfrak{g}}})$, when $\mathfrak{g}$ is of type ADE. We call $\mathfrak{K}_q(\mathfrak{g})$ the quantum virtual Grothendieck ring of $\mathfrak{g}$.

The main ingredient of the construction is the t-quantized Cartan matrix $\underline{\mathsf{C}}(t)$.

$$\mathsf{C}^{B_3}(q) = \begin{pmatrix} q^2+\frac{1}{q^2} & -1 & 0 \\ -1 & q^2+\frac{1}{q^2} & -1 \\ 0 & -q-\frac{1}{q} & q+\frac{1}{q} \end{pmatrix}, \quad \underline{\mathsf{C}}^{B_3}(t) = \begin{pmatrix} t+\frac{1}{t} & -1 & 0 \\ -1 & t+\frac{1}{t} & -1 \\ 0 & -2 & t+\frac{1}{t} \end{pmatrix}$$

Theorem (Kashiwara-O)

The Λ-invariants for the pairs of cuspidal modules $S_Q(\beta)$ and $S_Q(\gamma)$ over $R^{\mathfrak{g}}$ categorifying PBW-vectors adapted to Q in $\mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee}$ (including type $BCFG$) can be read from the inverse of $\underline{\mathsf{C}}(t)$.

Quantum virtual Grothendieck ring (very briefly)

Theorem (Jang-Lee-O)

(a) $\mathfrak{K}_q(\mathfrak{g})$ *has homogeneous bases* $\mathsf{F}_q = \{F_q(m)\}$, $\mathsf{E}_q = \{E_q(m)\}$ *and* $\mathsf{L}_q = \{L_q(m)\}$ *such that* (i) F_q *can be obtained by generalizing Frenkel-Mukhin-Hernandez algorithm via* $\underline{\mathsf{C}}(t)$, (ii) E_q *and* L_q *fits into the paradigm of Kazhdan-Lusztig theory:*

$$E_q(m) = L_q(m) + \sum_{m'} P_{m,m'}(q) L_q(m') \text{ for some } P_{m,m'}(q) \in q\mathbb{Z}[q].$$

(b) $\mathfrak{K}_q(\mathfrak{g})$ *has a quantum cluster algebra structure* $\mathscr{A}_q$ *of skew-symmetrizable type depending on* $\mathfrak{g}$.

The initial quantum cluster of $\mathfrak{K}_q(\mathfrak{g})$ consists of Kirillov-Reshetikhin(KR) polynomials and the quantum exchange relation we have applied corresponds to quantum folded T-system among KR-polynomials.

Quantum virtual Grothendieck ring 2 (very briefly)

Theorem (Kashiwara-O+Jang-Lee-O)

(c) *For a Dynkin quiver* $Q = (\blacktriangle, \xi)$, *we can define the heart subalgebra* $\mathfrak{K}_{q,Q}(\mathfrak{g})$ *and have an* $\mathbb{Z}[q^{\pm 1/2}]$*-algebra isomorphism*

$$\Psi_Q : K(R^{\mathfrak{g}}\text{-gmod}) \simeq \mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee} \xrightarrow{\sim} \mathfrak{K}_{q,Q}(\mathfrak{g}) \text{ sending}$$

1. *the the dual-canonical/upper-global basis of* $\mathcal{U}_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee}$ *to* $\mathsf{L}_{q,Q} := \mathsf{L}_q \cap \mathfrak{K}_{q,Q}(\mathfrak{g})$,
2. *the dual PBW-basis associated with reduced expressions of* w_0 *adapted to* Q *to* $\mathsf{E}_{t,\mathcal{Q}} := \mathsf{E}_t \cap \mathfrak{K}_{q,Q}(\mathfrak{g})$ *(up to* $\mathbb{Z}[q]^{\times}$*)*,

(d) *The* $\mathbb{Q}(q^{1/2})$*-algebra* $\mathbb{K}_q(\mathfrak{g}) := \mathbb{Q}(q^{1/2}) \otimes \mathfrak{K}_q(\mathfrak{g})$ *has presentation as follows:*

$$\sum_{s=0}^{1-\mathsf{c}_{i,j}} (-1)^s \begin{bmatrix} 1-\mathsf{c}_{i,j} \\ s \end{bmatrix}_{q_i} f_{i,k}^{1-\mathsf{c}_{i,j}-s} f_{j,k} f_{i,k}^{s} = 0,$$

$$f_{i,k} f_{j,k+1} = q^{-(\alpha_i,\alpha_j)} f_{j,k+1} f_{i,k} + (1 - q^{-(\alpha_i,\alpha_i)})\delta_{i,j},$$

$$f_{i,k} f_{j,l} = q^{(-1)^{k+l}(\alpha_i,\alpha_j)} f_{j,l} f_{i,k},$$

for $\mathsf{C} = (\mathsf{c}_{i,j})$ *of type* $\mathfrak{g}$, $i,j \in I$ *and* $k,l \in \mathbb{Z}$ *with* $l > k+1$.

THANK YOU