# Tensor Representations for the Drinfeld Double of the Taft Algebra

Benkart, Biswal, Kirkman, Nguyen Zhu

('87) A. Hopf alg $\underbrace{A \otimes A^*} = D_n$

↗ Taft

$\Bbbk = \overline{\Bbbk}$ $q \in \Bbbk$

$D_n$ $\langle a, b, c, d \mid \sim \rangle$

$q$: primative $\boxed{n}$-th root of unity

$bc - cb$

$ab - qba$, $a^n = d^n - 0$ $b^n - c^n - 1$,

$\underbrace{D_n} \twoheadrightarrow \underbrace{u_q(sl_2)}$

(Chen) known list of simples

· $\dim V(\ell, r) = \ell$ $1 \leq \ell \leq n$

$0 \leq r \leq n-1$

$\uparrow$

$\underline{P(\ell, r)} \twoheadrightarrow V(\ell, r)$

Fushion $\underbrace{V(\ell, r)} \otimes \underbrace{V(1, r')} = \underline{V(\ell, r + r')}$ ←

$D_n$ Quasitriangular (Drinfed '90)

$\exists$ $\underline{R} \in D_n \otimes D_n$

axioms V- $D_n$-mod

$\check{R} = \sigma R \curvearrowright \underline{V}^{\otimes 2}$ $\check{R} \to \mathrm{End}_{D_n}(V^{\otimes 2})$

↗ swap

(Ram-Leduk '97) $\check{R}_i \curvearrowright V \otimes \underbrace{\overset{i}{V} \otimes \overset{i+1}{V}} \otimes V = V^{\otimes k}$

· $\check{R}_i \check{R}_{i+1} \check{R}_i - \check{R}_{i+1} \check{R}_i \check{R}_{i+1}$

$$\Bbbk Br_k \longrightarrow \mathrm{End}_{D_n}(V^{\otimes k})$$

$$\left( \text{Jimbo '86} \quad \mathcal{H}_A^k \twoheadrightarrow \mathrm{End}_{U_q(sl_2)}(V^{\otimes k}) \right)$$

$$\underbrace{\mathcal{H}_A^k}\ := \Bbbk Br_k / \sim$$

$$Br_k := \langle s_i \mid 1 \le i \le k-1 \rangle$$

$$\sim : \quad \underline{(s_i - 1)(s_i + q^{-1}) = 0}$$

Th (BBKNZ) $\quad \underbrace{\mathcal{H}_A^k} \longrightarrow \mathrm{End}_{D_n}\left(\underbrace{V(2,r)}^{\otimes k}\right)$

$$s_i \longmapsto \underbrace{\lambda_r \check{R}_i}$$

$$\lambda_r = q^{-r(r+1)}$$

( Du-Parshall Scott '98
**Andersen**
Stroppel
- Tubbenhauer
'18) $\quad \underline{TL_k(\delta)} \xrightarrow{\sim} \mathrm{End}_{U_q(sl_2)}(V^{\otimes k})$ )

$$t_i = q^{1/2}(s_i - 1) \qquad \delta = -(q^{1/2} + q^{-1/2})$$

$$\underline{x^2 - q = 0}$$

$$\begin{cases} t_i^2 = \delta t_i \\ t_i t_j = t_j t_i \\ \underline{t_i t_{i+1} t_i - t_i} - t_{i+1} t_i t_{i+1} - t_{i+1} \quad (3) \end{cases}$$

$$TL_k(\delta) := \mathcal{H}_k^k / \sim$$

$$(3) = 0 \qquad \underline{t_i t_{i\pm1} t_i = t_i} \qquad \underline{V^{\otimes 3}}$$

$D_n$ is ribbon, when $n$ is odd

(Kaufman-Radford '93)

$$R = \sum_i x_i \otimes y_i$$

$$u = \sum_i y_i S(x_i)$$

$$c = u S(u) \leftarrow \text{central}$$

$c$ picks out simples summands of $W \otimes U$ by a scalar

ribbon $\Leftrightarrow$ $\exists$ $v$ central $v^2 = c$

Th (BBKNZ) $n$ is odd

$$v = u (bc)^{\frac{n-1}{2}}$$

in particular

$$v \curvearrowright V(l, r) \text{ by } q^{*}$$

$*$ ← dependent on $l, r$

Th $t_i t_{i\pm 1} t_i = t_i$ is satisfield

$$\boxed{\varphi \quad TL_k(q) \longrightarrow \operatorname{End}_{D_n}(V(2,r)^{\otimes k})}$$

$$t_i \longmapsto q^{1/2}(\lambda_r^{-1} \check{R}_i - \mathrm{id})$$

Th ~~f~~ $r = \frac{n-1}{2}$

$$V(2, \tfrac{n-1}{2}) \simeq \overset{*}{V}(2, \tfrac{n-1}{2})$$ natural mod for $U_q(sl_2)$

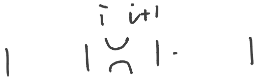
Th

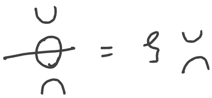
1) $\varphi$ is injective $\forall k$

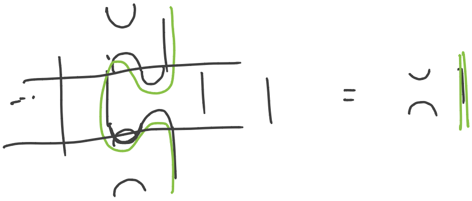
2) surjective $\forall\ 1 \le k \le 2(n-1)$

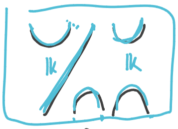
Remark Sharp bound !

# Injectivity

$t_i$ =









TL diag

noncrossing matching.

$$V \overset{f}{\cong} V^*$$

$$\begin{cases} \underline{ev.} \quad V \otimes V \xrightarrow{f \otimes id} V^* \otimes V \longrightarrow k \\ \qquad\qquad\qquad f \otimes v \longmapsto f(v) \\ \underline{coev} \quad k \longrightarrow V \otimes V^* \xrightarrow{id \otimes f^{-1}} V \otimes V \\ \qquad\quad 1 \longmapsto \underline{v_1} \otimes v_1^* + \underline{v_2} \otimes v_2^* \end{cases}$$

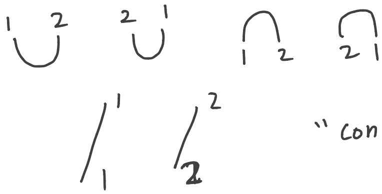
$$\underline{i} = (i_1, \cdots, i_k) \in \{1,2\}^k$$

$$v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k} \qquad D \quad \text{TL diag}$$

$D^{\underline{j}}_{\underline{i}}$ coeff of $\underline{v_{\underline{j}}}$ in the expansion of $\underline{D} \cdot \underline{v_{\underline{i}}}$

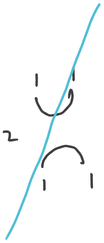
~~Th~~

<u>Th</u> $D^{\underline{j}}_{\underline{i}} = \begin{cases} \neq 0 & \text{iff} \\ 0 & \text{o.w.} \end{cases}$

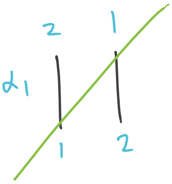



" consistent labeling

Eg / Proof when $k = 2$ ←

$E = \alpha_1$  $+ \ \alpha_2$  $= 0 \quad \in \mathrm{End}_{D_n}(V^{\otimes 2})$

$\overset{\bar{j}}{\underset{i}{E}} = 0 \quad \forall\, i, j$ $\qquad \alpha_1 = 0$

$\alpha_1$ 

$+ \ \alpha_2$ 

$\qquad \alpha_2 = 0$

Ehrig - Stroppel '16 $\qquad Br(\delta) \longrightarrow \mathrm{End}_{osp(n|2m+1)}(V^{\otimes k})$

$\Rightarrow \quad \underbrace{\dim V}_{\underbrace{\#\text{ labels}}_{\text{"}2\text{"}}} > \underbrace{k}_{\#\text{ strands}} \qquad k \to \infty$

"Proof": Inductive argument

induct on ~~an~~ partial order on TL - diag

( Russell - Tymoczko '19)

<u>Surjectivity</u>