

On τ -tilting finiteness of Schur algebras

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Introduction

As a generalization of the classical tilting modules, Adachi, Iyama and Reiten introduced the **support τ -tilting modules**, which have many nice properties. For example,

- these modules are bijectively corresponding to two-term silting complexes, left finite semibricks and so on.
- there is a partial order on the set of isomorphism classes of basic support τ -tilting modules such that its Hasse quiver realizes the left mutation. (See Proposition 1.4 in this talk.)

An algebra A over an algebraically closed field K is called **τ -tilting finite** if there are only finitely many pairwise non-isomorphic basic support τ -tilting A -modules. Otherwise, A is called **τ -tilting infinite**.

Background

(1) The τ -tilting finiteness for several classes of algebras is known, such as

- preprojective algebras of Dynkin type (Mizuno, 2014);
- algebras with radical square zero (Adachi, 2016);
- Brauer graph algebras (Adachi-Aihara-Chan, 2018);
- biserial algebras (Mousavand, 2019);
- minimal wild two-point algebras (W, 2019).

(2) Representation-finite algebras are τ -tilting finite and the converse is not true in general. For example, let

$$\Lambda_n := K(\circ \xrightarrow{a} \circ \curvearrowright b) / \langle b^n, ab^2 \rangle, \quad n \geq 2,$$

then Λ_n is τ -tilting finite (by direct calculation). But Λ_n is representation-finite if $n \leq 5$; tame if $n = 6$; wild if $n \geq 7$.

However, the converse is true in some special cases, including

- cycle-finite algebras (Malicki-Skowroński, 2016);
- gentle algebras (Plamondon, 2018);
- tilted and cluster-tilted algebras (Zito, 2019);
- simply connected algebras (W, 2019);
- quasi-tilted algebras, locally hereditary algebras, etc., (Aihara-Honma-Miyamoto-W, 2020).

The subject of this talk

Let n, r be positive integers and \mathbb{F} an algebraically closed field of characteristic p . We take an n -dimensional vector space V over \mathbb{F} with a basis $\{v_1, v_2, \dots, v_n\}$. Then, the r -fold tensor product $V^{\otimes r} := V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$ has a \mathbb{F} -basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leq i_j \leq n \text{ for all } 1 \leq j \leq r\}.$$

Let G_r be the symmetric group on r symbols. Then,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}.$$

for any $\sigma \in G_r$. We define the **Schur algebra** $S(n, r)$ by the endomorphism ring $\text{End}_{\mathbb{F}G_r}(V^{\otimes r})$.

In this talk, we will discuss the τ -tilting finiteness of $S(n, r)$, and see when the condition

$$\tau\text{-tilting finite} \Leftrightarrow \text{representation-finite}$$

is true on the class of Schur algebras.

Introduction

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τ -tilting theory

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Schur algebras

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Tame Schur algebras

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Wild Schur algebras

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References

τ -tilting theory

Auslander-Reiten translation

Let A be a finite-dimensional basic algebra over an algebraically closed field K . For an A -module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

the transpose of M is given by

$$\mathrm{Tr} M := \mathrm{coker} \mathrm{Hom}_A(d_1, A).$$

Then, the **Auslander-Reiten translation** is defined by

$$\tau(M) := D\mathrm{Tr} M,$$

where $D = \mathrm{Hom}_K(-, K)$ is the standard K -duality.

We denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M .

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right A -module and P a projective A -module.

1. M is called τ -rigid if $\text{Hom}_A(M, \tau M) = 0$.
2. M is called τ -tilting if M is τ -rigid and $|M| = |A|$.
3. M is called support τ -tilting if there exists an idempotent e of A such that M is a τ -tilting $(A/\langle e \rangle)$ -module.

We denote by τ -rigid A (resp., $s\tau$ -tilt A) the set of isomorphism classes of indecomposable τ -rigid (resp., basic support τ -tilting) A -modules.

Example

Let $A := K(1 \begin{smallmatrix} \xrightarrow{a} \\ \xleftarrow{b} \end{smallmatrix} 2) / \langle ab, ba \rangle$. We denote by S_i the simple A -modules and P_i the indecomposable projective A -modules. Then, we have

$$\tau(S_1) = S_2, \tau(S_2) = S_1, \tau(P_1) = 0, \tau(P_2) = 0.$$

Thus,

- τ -rigid $A = \{P_1, P_2, S_1, S_2, 0\}$;
- $s\tau$ -tilt $A = \{P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2, S_1, S_2, 0\}$;
- $P_1 \oplus P_2, P_1 \oplus S_1$ and $S_2 \oplus P_2$ are τ -tilting modules.

We denote by $\text{add}(M)$ (resp., $\text{Fac}(M)$) the full subcategory whose objects are direct summands (resp., factor modules) of finite direct sums of copies of M .

A morphism $\pi : M \rightarrow N'$ is called a **minimal left $\text{add}(N)$ -approximation of M** if $N' \in \text{add}(N)$ and it satisfies:

(1) any $h : N' \rightarrow N'$ satisfying $h \circ \pi = \pi$ is an automorphism.

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N' \\ & \searrow \pi & \downarrow h \simeq \text{id} \\ & & N' \end{array}$$

(2) for any $N'' \in \text{add}(N)$ and $g : M \rightarrow N''$, there exists $f : N' \rightarrow N''$ such that $f \circ \pi = g$.

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N' \\ & \searrow \forall g & \downarrow \exists f \\ & & N'' \end{array}$$

Mutation

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

Let $T = M \oplus N$ be a basic support τ -tilting A -module with an indecomposable direct summand M satisfying $M \notin \text{Fac}(N)$. We take a minimal left $\text{add}(N)$ -approximation π with an exact sequence

$$M \xrightarrow{\pi} N' \longrightarrow \text{coker } \pi \longrightarrow 0.$$

Then, we call $\mu_M^-(T) := \text{coker } \pi \oplus N$ the **left mutation of T with respect to M** , which is again a basic support τ -tilting A -module.

Example

Let $A = K(1 \begin{smallmatrix} \xrightarrow{a} \\ \xleftarrow{b} \end{smallmatrix} 2) / \langle ab, ba \rangle$, then $P_1 \oplus P_2$ is a τ -tilting module. We consider the left mutation with respect to P_2 ,

$$P_2 \xrightarrow{\pi} P_1 \longrightarrow \operatorname{coker} \pi \longrightarrow 0,$$

where $\pi : \begin{smallmatrix} e_2 \\ b \end{smallmatrix} \xrightarrow{a} e_1$ is a minimal left $\operatorname{add}(P_1)$ -approximation. Then,

$$\operatorname{coker} \pi = S_1 \text{ and } \mu_{P_2}^-(A) = P_1 \oplus S_1.$$

In fact, we have the following **mutation quiver** of $s\tau$ -tilt A .

$$\begin{array}{ccccc} P_1 \oplus P_2 & \longrightarrow & P_1 \oplus S_1 & \longrightarrow & S_1 \\ \downarrow & & & & \downarrow \\ S_2 \oplus P_2 & \longrightarrow & S_2 & \longrightarrow & 0 \end{array}$$

Poset structure on $s\mathcal{T}$ -tilt A

Definition 1.3 (Adachi-Iyama-Reiten, 2014)

For $M, N \in s\mathcal{T}\text{-tilt } A$, we say $M \geq N$ if $\text{Fac}(N) \subseteq \text{Fac}(M)$.

Example

Let A be the previous algebra. The [Hasse quiver](#) of $s\mathcal{T}$ -tilt A is

$$\begin{array}{ccccc}
 P_1 \oplus P_2 & \xrightarrow{>} & P_1 \oplus S_1 & \xrightarrow{>} & S_1 \\
 \downarrow > & & & & \downarrow > \\
 S_2 \oplus P_2 & \xrightarrow{>} & S_2 & \xrightarrow{>} & 0
 \end{array}$$

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

The mutation quiver $\mathcal{Q}(s_{\mathcal{T}}\text{-tilt } A)$ and the Hasse quiver $\mathcal{H}(s_{\mathcal{T}}\text{-tilt } A)$ coincide.

Proposition 1.5 (Adachi-Iyama-Reiten, 2014)

If the Hasse quiver $\mathcal{H}(s_{\mathcal{T}}\text{-tilt } A)$ contains a finite connected component, then it exhausts all pairwise non-isomorphic basic support τ -tilting A -modules.

Reduction theorems

Proposition 1.6 (Demonet-Iyama-Jasso, 2017)

If A is τ -tilting finite, then

- (1) the quotient A/I is τ -tilting finite for any two-sided ideal I of A .
- (2) the idempotent truncation eAe is τ -tilting finite for any idempotent e of A .

Proposition 1.7 (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by central elements which are contained in the Jacobson radical of A . Then, there exists a poset isomorphism between $s\tau$ -tilt A and $s\tau$ -tilt (A/I) .

Schur algebras

Permutation modules

Let \mathbb{F} be an algebraically closed field of characteristic p .

- r : a positive number.
- $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$: a partition of r with at most n parts.
- $\Omega(n, r)$: the set of all partitions of r with at most n parts.
- G_r : the symmetric group on r symbols.
- $\mathbb{F}G_r$: the group algebra of G_r over \mathbb{F} .

We may define the Young subgroup G_λ of G_r associated with a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of r by

$$G_\lambda := G_{\lambda_1} \times G_{\lambda_2} \times \dots \times G_{\lambda_n}.$$

Then, the **permutation $\mathbb{F}G_r$ -module** M^λ is $1_{G_\lambda} \uparrow^{G_r}$, where 1_{G_λ} denotes the trivial module for G_λ and \uparrow denotes the induction.


Let $S(n, r) = \text{End}_{\mathbb{F}G_r}(V^{\otimes r})$ be the Schur algebra. We have the following well-known algebra isomorphism

$$S(n, r) \simeq \text{End}_{\mathbb{F}G_r} \left(\bigoplus_{\lambda \in \Omega(n, r)} n_\lambda M^\lambda \right),$$

where $1 \leq n_\lambda \in \mathbb{N}$ is the number of compositions of r with at most n parts which are rearrangement of λ .

Specht modules and the decomposition matrix

Let p be a prime and $[\lambda]$ the Young diagram of partition λ . We call λ a **p -regular** partition if no p rows of $[\lambda]$ have the same length. Otherwise, λ is called p -singular.

A 3-regular partition: $[(2, 2, 1)] =$


Let S^λ be the **Specht module** of $\mathbb{F}G_r$ corresponding to λ .

- If $\text{char } \mathbb{F} = 0$, then $\{S^\lambda \mid \lambda : \text{partition of } r\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$ -modules.
- If $\text{char } \mathbb{F} = p$, then $\{D^\lambda \mid \lambda : p\text{-regular partition of } r\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$ -modules, where D^λ is the unique simple top $S^\lambda / (\text{rad } S^\lambda)$ of S^λ with λ being p -regular.

In the case of a p -singular partition μ , all of the composition factors of S^μ are D^λ such that λ is a p -regular partition which dominates μ . Therefore, the **decomposition matrix** of $\mathbb{F}G_r$ has the following form (see [James, 1978]),

$$\begin{array}{l}
 S^\lambda, \lambda \text{ } p\text{-regular} \\
 S^\lambda, \lambda \text{ } p\text{-singular}
 \end{array}
 \left\{ \begin{array}{c}
 \overbrace{\left(\begin{array}{cccc}
 1 & & & \\
 * & 1 & & 0 \\
 * & * & 1 & \\
 \vdots & \vdots & \vdots & \ddots \\
 * & * & * & \dots & 1 \\
 \hline
 * & * & * & \dots & * \\
 * & * & * & \dots & *
 \end{array} \right)}^{D^\mu, \mu \text{ } p\text{-regular}}
 \end{array} \right. .$$

Young modules

Let \mathbb{F} be an algebraically closed field and p a prime.

- χ^λ : the ordinary character corresponding to Specht module S^λ over $\text{char } \mathbb{F} = 0$.
- $\text{ch } M^\lambda$: the associated ordinary character of permutation module M^λ over $\text{char } \mathbb{F} = p$.

Then, χ^λ is a constituent of $\text{ch } M^\lambda$ and χ^μ ($\mu \neq \lambda$) is a constituent of $\text{ch } M^\lambda$ if and only if $\mu \triangleright \lambda$.

Let $M^\lambda := \bigoplus_{i=1}^n Y_i$ with $n \in \mathbb{N}$. The unique direct summand Y_i which the ordinary character χ^λ occurs in $\text{ch } Y_i$, is called the **Young module** corresponding to λ and we denote it by Y^λ .

Let Y^λ be the Young module corresponding to λ .

- $Y^\lambda = (Y^\lambda)^*$ with $(-)^* := \text{Hom}(-, \mathbb{F})$.
- Y^λ has a Specht filtration: $Y^\lambda = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k = 0$ for some $k \in \mathbb{N}$ with each Z_i/Z_{i+1} isomorphic to a Specht module S^μ with $\mu \trianglerighteq \lambda$.
- If λ is a partition with at most n parts, then each composition factor D^μ of Y^λ is also corresponding to the partition μ with at most n parts.

Proposition 2.1 (Martin, 1993)

The set $\{Y^\lambda \mid \lambda \in \Omega(n, r)\}$ is a complete set of pairwise non-isomorphic Young modules which occurs as indecomposable direct summands of permutation modules M^λ with $\lambda \in \Omega(n, r)$.

The basic algebra of $S(n, r)$

Let B be a block of $\mathbb{F}G_r$ labeled by a p -core ω , it is well-known that a partition λ belongs to B if and only if λ has the same p -core ω . Then, we define

$$S_B := \text{End}_{\mathbb{F}G_r} \left(\bigoplus_{\lambda \in B \cap \Omega(n, r)} Y^\lambda \right)$$

and the **basic algebra** of $S(n, r)$ is $\bigoplus S_B$, where the sum is taken over all blocks of $\mathbb{F}G_r$. Moreover, S_B is a direct sum of blocks of the basic algebra of $S(n, r)$.

Reduction theorems on $S(n, r)$

Lemma 2.2 (W, 2020)

If $S(n, r)$ is τ -tilting infinite, then so is $S(N, r)$, for any $N > n$.

Sketch of the proof: We show that $S(n, r)$ is an idempotent truncation of $S(N, r)$.

Lemma 2.3 (W, 2020)

If $S(n, r)$ is τ -tilting infinite, then so is $S(n, n + r)$.

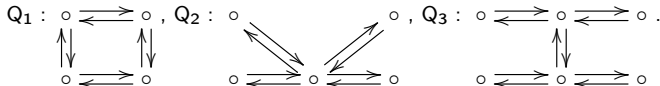
Sketch of the proof: It is shown by [Erdmann, 1993] that $S(n, r)$ is a quotient of $S(n, n + r)$.

Strategy on τ -tilting infinite Schur algebras

Let $A := \mathbb{F}Q/I$ be an algebra. We call Q a τ -tilting infinite quiver if $A/\text{rad}^2 A$ is τ -tilting infinite. For example, the Kronecker quiver $Q : \circ \rightrightarrows \circ$ is a τ -tilting infinite quiver.

Lemma 2.4 (W, 2020)

The following Q_1, Q_2 and Q_3 are τ -tilting infinite quivers.



Sketch of the proof: We remark that representation-infinite path algebras are τ -tilting infinite.

Useful information on $S(2, r)$

A partition (λ_1, λ_2) of r is uniquely determined by $s = \lambda_1 - \lambda_2$.

- v^s : the vertex in the quiver of the basic algebra of $S(2, r)$ corresponding to the Young module $Y^{(\lambda_1, \lambda_2)}$.
- $n(v^s, v^t)$: the number of arrows from v^s to v^t .

Theorem 2.5 (Erdmann-Henke, 2002)

We have $n(v^s, v^t) = n(v^t, v^s)$. Let $s = s_0 + ps'$ and $t = t_0 + pt'$ with $0 \leq s_0, t_0 \leq p - 1$ and $s', t' \geq 0$. Suppose $s > t$.

(1) If $p = 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0 = 1 \text{ or } s_0 = t_0 = 0 \text{ and } s' \equiv t' \pmod{2}, \\ 1 & \text{if } s_0 = t_0 = 0, t' + 1 = s' \not\equiv 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) If $p > 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0, \\ 1 & \text{if } s_0 + t_0 = p - 2, t' + 1 = s' \not\equiv 0 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1

Let $p = 2$. We consider the quiver of the basic algebra of $S(2, 10)$. By Theorem 2.5, we have

$$\begin{array}{ccccc}
 & & (6, 4) & \rightleftharpoons & (10) & . \\
 & & \uparrow \downarrow & & \uparrow \downarrow & \\
 (8, 2) & \rightleftharpoons & (7, 3) & \rightleftharpoons & (5^2) & \rightleftharpoons & (9, 1)
 \end{array}$$

It is obvious that $S(2, 10)$ is τ -tilting infinite.

Example 2

Let $p = 2$. The quiver of the basic algebra of $S(2, 11)$ is

$$(10, 1) \rightleftharpoons (6, 5) \rightleftharpoons (8, 3) \quad (11) \rightleftharpoons (7, 4) \quad (9, 2) .$$

We have to find the explicit structures of Young modules.

Let p be a prime. For any non-negative integer s , there is a p -adic decomposition $s = \sum_{k \geq 0} s_k p^k$. Now, let s, t be two non-negative integers, we define a function

$$f(s, t) = \prod_{k \in \{0\} \cup \mathbb{N}} \binom{p-1-s_k}{p-1-t_k}.$$

Moreover, we have

$$g(s, t) := \begin{cases} 1 & \text{if } f(2t, s+t) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$h(s, t) := \begin{cases} 1 & \text{if } f(2t+1, s+t+1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6 (Henke, 1999)

Let $(r - k, k)$ be a partition with a non-negative integer k .

(1) If r is even, then

$$\text{ch } Y^{(r-k, k)} = \sum_{i=0}^{\frac{r}{2}} g\left(\frac{r}{2} - i, \frac{r}{2} - k\right) \chi^{(r-i, i)}.$$

(2) If r is odd, then

$$\text{ch } Y^{(r-k, k)} = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} h\left(\lfloor \frac{r}{2} \rfloor - i, \lfloor \frac{r}{2} \rfloor - k\right) \chi^{(r-i, i)},$$

where $\lfloor \frac{r}{2} \rfloor$ is the greatest integer less than $\frac{r}{2}$.

Suppose $p = 2$. Let B_1 be the principal block of $\mathbb{F}G_{11}$ and B_2 the block of $\mathbb{F}G_{11}$ labeled by 2-core $(2, 1)$, the parts of the decomposition matrix $[S^\lambda : D^\mu]$ (see [James, 1978]) for the partitions in B_1 and B_2 with at most two parts are

$$B_1 : \begin{matrix} (11) \\ (9, 2) \\ (7, 4) \end{matrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix}, \quad B_2 : \begin{matrix} (10, 1) \\ (8, 3) \\ (6, 5) \end{matrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}.$$

We first determine S_{B_2} . By Theorem 2.6, we have

$$\begin{aligned} \text{ch } Y^{(10,1)} &= \chi^{(10,1)}, \\ \text{ch } Y^{(8,3)} &= \chi^{(10,1)} + \chi^{(8,3)}, \\ \text{ch } Y^{(6,5)} &= \chi^{(10,1)} + \chi^{(8,3)} + \chi^{(6,5)}. \end{aligned}$$

Recall that $\dim \text{Hom}_{\mathbb{F}G_{11}}(Y^\lambda, Y^\mu)$ is equal to the inner product $(\text{ch } Y^\lambda, \text{ch } Y^\mu)$. Then, we have

- $Y^{(10,1)} = D^{(10,1)}$, $Y^{(8,3)} = \begin{matrix} D^{(10,1)} \\ D^{(8,3)} \end{matrix}$,

- $Y^{(6,5)} = \begin{matrix} & D^{(8,3)} & & D^{(10,1)} \\ & / \quad \backslash & & / \\ D^{(10,1)} & & D^{(6,5)} & & D^{(8,3)} \end{matrix}$.

Hence, $S_{B_2} = \text{End}_{\mathbb{F}G_{11}}(Y^{(10,1)} \oplus Y^{(8,3)} \oplus Y^{(6,5)})$ is isomorphic to $\mathbb{F}Q/I$ with

$$Q : (10, 1) \begin{matrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{matrix} (6, 5) \begin{matrix} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{matrix} (8, 3) \text{ and} \\ I : \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

Similarly, $S_{B_1} = \mathbb{F} \left(\begin{matrix} (11) \begin{matrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{matrix} (7, 4) & (9, 2) \end{matrix} \right) / \langle \alpha_1\beta_1 \rangle.$

We point out the following facts.

- S_B is uniquely determined by the decomposition matrix $[S^\lambda : D^\mu]$ and the character $\text{ch } Y^\lambda$ of Young modules;
- We may compute the basic algebra of $S(2, r)$ by hand.
- We may compute the basic algebra of $S(n, r)$ with $n \geq 3$ by a computer. For example, we refer to Carlson and Matthews's program:

<http://alpha.math.uga.edu/jfc/schur.html>.

Tame Schur algebras

The complete classification of representation type

The representation type of Schur algebras is completely determined by [Erdmann, 1993], [Xi, 1993], [Doty-Nakano, 1998] and [Doty-Erdmann-Martin-Nakano, 1999].

Proposition 2.7

Let p be the characteristic of \mathbb{F} . Then, $S(n, r)$ is

- semi-simple if and only if $p = 0$ or $p > r$ or $p = 2, n = 2, r = 3$;
- representation-finite if and only if $p = 2, n = 2, r = 5, 7$ or $p \geq 2, n = 2, r < p^2$ or $p \geq 2, n \geq 3, r < 2p$;
- infinite-tame if and only if $p = 2, n = 2, r = 4, 9, 11$ or $p = 3, n = 2, r = 9, 10, 11$ or $p = 3, n = 3, r = 7, 8$.

Otherwise, $S(n, r)$ is wild.

Representation-finite blocks of Schur algebras

Proposition 2.8 (Erdmann, 1993; Donkin-Reiten, 1994)

Let A be a representation-finite block of Schur algebras, then it is Morita equivalent to $\mathcal{A}_m := \mathbb{F}Q/I$ for some $m \in \mathbb{N}$, where

$$Q : 1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \cdots \begin{array}{c} \xrightarrow{\alpha_{m-2}} \\ \xleftarrow{\beta_{m-2}} \end{array} m-1 \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} m ,$$

$$I : \langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \leq i \leq m-2 \rangle.$$

Theorem 2.9 (W, 2020)

We have $\#\text{s}\tau\text{-tilt } \mathcal{A}_m = \binom{2m}{m}$.

Sketch of the proof: We remark that \mathcal{A}_m is a quotient of a Brauer line algebra modulo the two-sided ideal generated by the central element $\alpha_1\beta_1$. Then, the number follows from the result by [Aoki, 2018].

Tame blocks of tame Schur algebras

Proposition 2.10 (Doty-Erdmann-Martin-Nakano, 1999)

Let A be an infinite-tame block of tame Schur algebras, then it is Morita equivalent to one of \mathcal{D}_3 , \mathcal{D}_4 , \mathcal{R}_4 and \mathcal{H}_4 , where

- Let $\mathcal{D}_3 := \mathbb{F}Q/I$ be the special biserial algebra given by

$$Q : \circ \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \circ \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \circ \quad \text{and } I : \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

- Let $\mathcal{D}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q : \circ \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1\alpha_2} \end{array} \circ \begin{array}{c} \xrightarrow{\beta_3} \\ \xleftarrow{\alpha_3} \\ \downarrow \beta_2 \\ \circ \end{array} \circ \quad \text{and } I : \left\langle \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_1, \alpha_3\beta_2, \alpha_1\beta_3, \alpha_2\beta_3, \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \beta_2\alpha_2 - \beta_3\alpha_3 \right\rangle.$$

- Let $\mathcal{R}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q : \circ \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \circ \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \circ \begin{array}{c} \xrightarrow{\alpha_3} \\ \xleftarrow{\beta_3} \end{array} \circ \quad \text{and } I : \left\langle \alpha_1\beta_1, \alpha_1\alpha_2, \beta_2\beta_1, \alpha_2\beta_2 - \beta_1\alpha_1, \alpha_3\beta_3 - \beta_2\alpha_2 \right\rangle.$$

- Let $\mathcal{H}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q : \begin{array}{ccccc} \circ & \xrightarrow{\alpha_1} & \circ & \xrightarrow{\alpha_3} & \circ \\ & \beta_1 \swarrow & \downarrow \beta_2 & \nwarrow \beta_3 & \\ & \alpha_2 \searrow & \circ & & \\ & & & & \circ \end{array} \text{ and } I : \left\langle \begin{array}{l} \alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2, \alpha_1\alpha_3, \\ \beta_3\beta_1, \alpha_3\beta_3 - \beta_1\alpha_1 - \beta_2\alpha_2 \end{array} \right\rangle.$$

Lemma 2.11 (W, 2020)

The algebras \mathcal{D}_3 , \mathcal{D}_4 , \mathcal{R}_4 and \mathcal{H}_4 are τ -tilting finite. Moreover,

A	\mathcal{D}_3	\mathcal{D}_4	\mathcal{R}_4	\mathcal{H}_4
$\#_{s\tau\text{-tilt}} A$	28	114	88	96

Sketch of the proof: We show this lemma by direct computation with some reduction steps and software.

Theorem 2.12 (W, 2020)

If the Schur algebra $S(n, r)$ is tame, then it is τ -tilting finite.

Sketch of the proof: We have showed that \mathcal{A}_m , \mathcal{D}_3 , \mathcal{D}_4 , \mathcal{R}_4 and \mathcal{H}_4 are τ -tilting finite. It suffices to make clear that these are all the blocks of tame Schur algebras. For example, let $p = 3$, then

- the basic algebra of $S(2, 9)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F}$;
- the basic algebra of $S(2, 10)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2, 11)$ is isomorphic to $\mathcal{D}_4 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 7)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 8)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{H}_4 \oplus \mathcal{A}_2$.

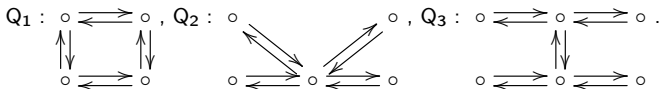
Wild Schur algebras

We consider wild Schur algebras except for the following cases.

$$(\star) \begin{cases} p = 2, n = 2, r = 8, 17, 19; \\ p = 2, n = 3, r = 4; \\ p = 2, n \geq 5, r = 5; \\ p \geq 5, n = 2, p^2 \leq r \leq p^2 + p - 1. \end{cases}$$

A quick review:

- $S(n, r)$ τ -tilting infinite
 $\Rightarrow S(N, r)$ ($N > n$) τ -tilting infinite
 $\Rightarrow S(n, n + r)$ τ -tilting infinite
- The following Q_1, Q_2 and Q_3 are τ -tilting infinite quivers.



The characteristic $p = 2$

Blue: τ -tilting finite **Red:** τ -tilting infinite

r	1	2	3	4	5	6	7	8	9	10	11	12
$S(2, r)$	S	F	S	T	F	W	F	W	T	W	T	W
r	13	14	15	16	17	18	19	20	21	22	23	...
$S(2, r)$	W	W	W	W	W	W	W	W	W	W	W	...

Sketch of the proof: We show the following.

- the basic algebra $\overline{S(2, 6)}$ of $S(2, 6)$ is τ -tilting finite;
- the basic algebra of $S(2, 10)$ is τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(2, 13)$ is $\overline{S(2, 6)} \oplus \mathcal{A}_2 \oplus \mathbb{F}$;
- the basic algebra of $S(2, 15)$ is $\overline{S(2, 6)} \oplus \mathcal{A}_2 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2, 21)$ is τ -tilting infinite ($\Leftarrow Q_1$).

The characteristic $p = 2$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	...
3	S	F	F	W	W	W	W	W	W	W	W	W	W	...
4	S	F	F	W	W	W	W	W	W	W	W	W	W	...
5	S	F	F	W	W	W	W	W	W	W	W	W	W	...
6	S	F	F	W	W	W	W	W	W	W	W	W	W	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Sketch of the proof:

- the basic algebra of $S(3, 6)$ is τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(3, 7)$ is τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(3, 8)$ is τ -tilting infinite ($\Leftarrow Q_2$);
- the basic algebra of $S(4, 4)$ is τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(4, 5)$ is τ -tilting finite.

The characteristic $p = 3$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	...
2	S	S	F	F	F	F	F	F	T	T	T	W	W	...
3	S	S	F	F	F	W	T	T	W	W	W	W	W	...
4	S	S	F	F	F	W	W	W	W	W	W	W	W	...
5	S	S	F	F	F	W	W	W	W	W	W	W	W	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Sketch of the proof:

- the basic algebras of $S(2, 12)$ and $S(2, 13)$ are τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(3, 6)$ is τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebras of $S(3, 10)$ and $S(3, 11)$ are τ -tilting infinite ($\Leftarrow Q_3$);
- the basic algebra of $S(4, 7)$ is τ -tilting infinite ($\Leftarrow Q_2$);
- the basic algebra of $S(4, 8)$ is τ -tilting infinite ($\Leftarrow Q_1$).

The characteristic $p \geq 5$

$n \backslash r$	$1 \sim p-1$	$p \sim 2p-1$	$2p \sim p^2-1$	$p^2 \sim p^2+p-1$	$p^2+p \sim \infty$
2	S	F	F	W	W
3	S	F	W	W	W
4	S	F	W	W	W
5	S	F	W	W	W
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Sketch of the proof:

- the basic algebras of $S(2, p^2 + p)$ and $S(2, p^2 + p + 1)$ are τ -tilting infinite ($\Leftarrow Q_1$);
- the basic algebras of $S(3, 2p)$, $S(3, 2p + 1)$ and $S(3, 2p + 2)$ are τ -tilting infinite ($\Leftarrow Q_3$).

Theorem 2.13 (W, 2020)

Let $p \geq 5$ and except for the cases in (\star) , the Schur algebra $S(n, r)$ satisfies the condition:

$$\tau\text{-tilting finite} \Leftrightarrow \text{representation-finite}$$

Theorem 2.14 (W, 2020)

Let $p = 3$, the Schur algebra $S(n, r)$ satisfies the condition:

$$\tau\text{-tilting infinite} \Leftrightarrow \text{wild.}$$

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Thank you very much for your attention !