On Kazhdan–Lusztig cells of $\mathfrak{a}$-value 2

(Joint work with Richard Green)

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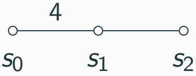
Kazhdan–Lusztig Cells of a-value 2

Coxeter groups

- *Coxeter groups* are groups with special presentations encoded by weighted graphs; the graphs are called *Coxeter digrams*.



$m_{s_0 s_1} = 4$, $m_{s_1 s_2} = 3$

$m_{s_0 s_2} = 2$

$\rightsquigarrow$

$$W = \langle S \mid R \rangle, \quad S = \{s_0, s_1, s_2\},$$

$$\text{relations: } s_0^2 = s_1^2 = s_2^2 = 1,$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0,$$

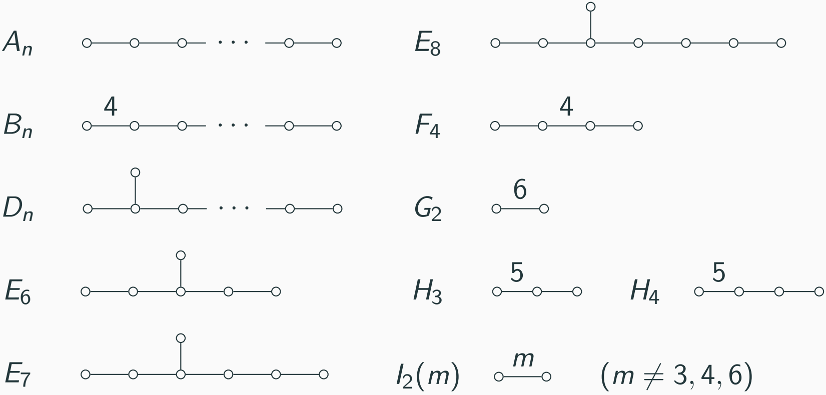
$$s_1 s_2 s_1 = s_2 s_1 s_2, \; s_0 s_2 = s_2 s_0.$$

We call each relation ($sts \cdots = tst \cdots$) a *braid relation*.
If $m(s,t) = 2$, we also call ($st = ts$) a *commutation relation*.

- The above group W is the hyperoctahedral group B_3; the subgroup $\langle s_1, s_2 \rangle \subseteq W$ is the symmetric group A_2. Indeed...

Classification of finite Coxeter groups

All Weyl and affine Weyl groups are Coxeter groups, and finite Coxeter systems have a well-known classification:



Classification of finite irreducible Coxeter systems.

Hecke algebras

- Each Coxeter system (W, S) gives rise to a *Hecke algebra* H, an associative algebra over the ring $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$ generated by a set $\{T_s : s \in S\}$ subject to the relations

$$(T_s - v)(T_s + v^{-1}) = 0 \quad \forall s \in S,$$

$$\underbrace{T_s T_t T_s \cdots = T_t T_s T_t \cdots}_{m_{s,t} \text{ factors on each side}} \quad \forall s, t \in S.$$

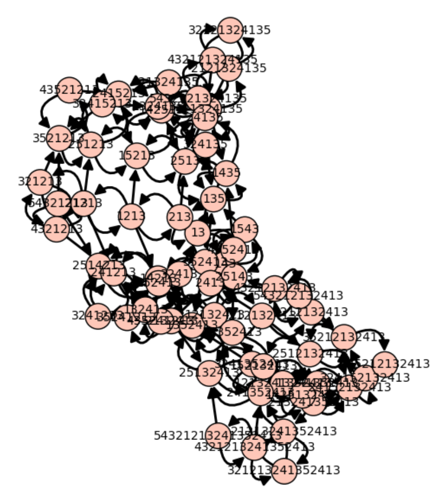
  The algebra H recovers the group algebra $\mathbb{Z}W$ when $v = 1$.
- The algebra H has a *Kazhdan–Lusztig* (*KL*) *basis* $\{C_w : w \in W\}$, which has remarkable properties and gives rise to both the KL cells of W and the a-function on W.

Kazhdan–Lusztig cells

- We can define an equivalence relation $\sim_L$ on W as follows:
  1. define $x \prec_L y$ if C_x appears in $C_s C_y$ for some $s \in S$;
  2. let $\leq_L$ be the transitive and reflexive closure of $\prec_L$;
  3. declare $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$.

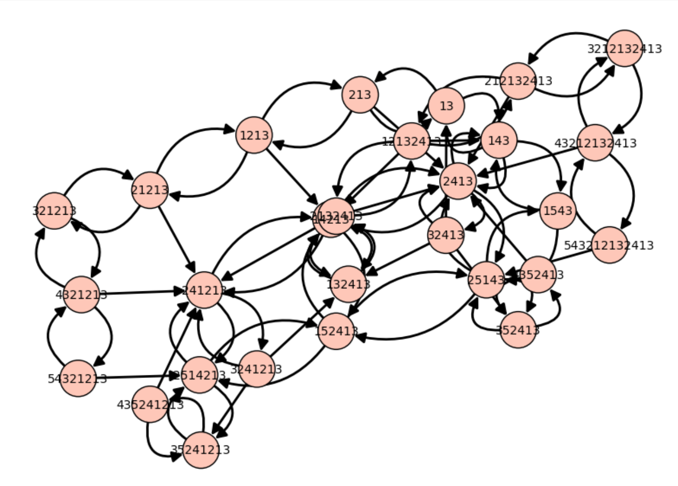
  The equivalence classes of $\sim_L$ are the *left KL cells* of W. *Right cells* can be defined similarly.
- So can *two-sided cells*: use "$x \prec_{LR} y$ if $x \prec_L y$ or $x \prec_R y$". It follows that each two-sided cell is a union of left cells and also of right cells.
- For a right cell R, the set $R^{-1} = \{w^{-1} : w \in R\}$ is always a left cell. Similarly, setwise inverses of left cells are right cells.

Example: computation of a left cell in type H_5



The set $\{w \in H_5 : w \leq_L 13\}$.

Example: computation of a left cell in type H_5



The left cell $L = \{w \in H_5 : w \leq_L 13 \leq_L w\}$ of 13.

Cell modules

- Each left cell L of W induces a *cell module* of H defined as

$$H_\Gamma := \left(\oplus_{y \leq_L \Gamma} \mathcal{A} C_y\right) / \left(\oplus_{y <_L \Gamma} \mathcal{A} C_y\right).$$

  Here H_Γ has a basis indexed by Γ, and H acts on H_Γ by multiplication.

- **Example:** In type A, KL cells can be determined by the Robinson–Schensted correspondence by Kazhdan–Lusztig and Ariki; left cell modules are naturally isomorphic to the Specht modules of H (Dipper–James) by McDonough–Pallikaros.

The a-function

Proposition (Lusztig)

Suppose that

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \quad \forall x, y \in W.$$

1. For each $z \in W$, there **exists** a unique integer $\mathrm{a}(z) \geq 0$ s.t.

$$\mathrm{a}(z) = \max\{\deg h_{x,y,z} : x, y \in W\}.$$

2. For $x, y \in W$, $\mathrm{a}(x) = \mathrm{a}(y)$ if $x \sim_{LR} y$.

Remarks on the a-function

- For Weyl groups, the a-function measures the Gelfand–Kirillov co-dimensions of certain highest weight modules of the corresponding Lie algebras.
- For affine Weyl groups, the a-function measures the dimensions of varieties of certain Borel subgroups of reductive algebraic groups.
- According to Lusztig, the construction of KL cells and a-functions from KL bases can be generalized to many quantum algebras with "nice" bases, including modified quantum enveloping algebras, q-Schur algebras and Temperley–Lieb algebras.

Cells with low a-values

Theorem (Lusztig 1984, Hart 2017)

Let W be an irreducible Coxeter group, and let $w \in W$. Then

- We have $\mathrm{a}(w) = 0$ iff $w = 1_W$;
- We have $\mathrm{a}(w) = 1$ iff w has a unique non-empty reduced word. Such elements form a single two-sided cell, with the elements whose unique reduced words begin in s forming a single right cell R_s for each $s \in S$.
- W has finitely many elements of a-value 1 iff its Coxeter diagram is a tree with at most one edge of weight at least 4. When this is the case, we can count the elements of a-value 1.

Our goal is to extend these results to a-value 2. Specifically . . .

Summary of results

Let $W_2 = \{w \in W : \mathrm{a}(w) = 2\}$.

(1) We can characterize when W_2 is finite via Coxeter diagrams.

(2) We have explicit combinatorial descriptions of the left, right, and two-sided cells in W_2, as well as their intersections, whenever W_2 is finite. We can also count all cells in W_2.

(3) Moreover, we can do (2) in a uniform way via what we call *stubs*, without using recursions involving the KL basis.

(4) Let V be the reflection representation of W. In types E_{pqr}, we can use stubs to relate the cell module of any left cell of a-value 2 with a certain codimension-1 submodule of $S^2(V)$.

The starting point of our results is the following observation:

Proposition (Green–X. 2018)

Let $w \in W$. If $\mathrm{a}(w) = 2$, then w is fully commutative.

Fully Commutative Elements

Full commutativity

An element $w \in W$ is called *fully commutative* (FC) if all its reduced words can be connected via only commutation relations.

Proposition (Stembridge 1996)

An element w is fully commutative if and only if no reduced word of w contains a contiguous subword $sts\dots$ of length $m(s,t)$ where $s, t \in S$ and $m(s,t) \geq 3$.

Example

In type B_3, with $S = \{1, 2, 3\}$ and $m(1, 2) = 4$, the element $12132 = 12312$ is FC but the element $13212 = 31212$ is not.

FC elements are connected to Schubert polynomials, Catalan combinatorics, Temperley–Lieb algebras and quiver Hecke algebras.

Heaps

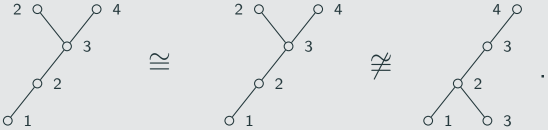
The main tool we will use for FC elements is heaps.

- Each word $\underline{w} \in S^*$ gives rise to a heap poset:

Example

In type A_4, with $S = \{1, 2, 3, 4\}$ and $m(i, i+1) = 3$ for each i,

$$12342 \overset{2,4}{=} 12324 \overset{2,3}{=} 13234\ ,$$



- If w is FC, then all reduced words of w give rise to the same heap; we call it the *heap* of w and denote it by $H(w)$.

Heap criterion for full commutativity

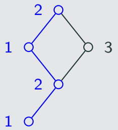
Proposition (Stembridge 1996)

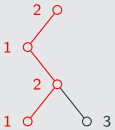
A word $\underline{w}$ is the reduced word of an FC element if and only if $H(\underline{w})$ satisfies the following conditions:

- No column contains two points connected by an edge.
- For any $s, t \in S$ with $m(s,t) \geq 3$, there is no **convex** chain of edges connecting a sequence $s, t, \cdots$ of $m(s,t)$ points.

Example

The element 12132 is FC in B_3 while 13212 is not:





Descents via heaps

- Each $w \in W$ has a *left descent* set

$$\mathcal{L}(w) = \{s \in S : l(sw) < l(w)\}.$$

Example: For the element $w = 1(232)4 = (13)234 = 31234$ in A_4, we have $\{1, 3\} \subseteq \mathcal{L}(w)$ as the above words are reduced.

- It is known that $\mathcal{L}(x) = \mathcal{L}(y)$ if $x \sim_R y$.
- If w is FC, then $s \in \mathcal{L}(w)$ if and only if in some reduced word $\underline{w}$ of w, all letters to the left of s commute with s. Thus, $\mathcal{L}(w)$ consists of the minimal ("bottom") elements in $H(w)$.

Obvious counterparts of the above facts exist for right descents.

a-values via heaps

We can often use heaps to compute the a-function:

Theorem (Shi 2005)

Let W be a Coxeter group. Let $w \in W$ be FC. Let $n(w)$ be the maximum cardinality of an antichain in $H(w)$. Then

- $\mathrm{a}(w) \geq n(w)$.
- If W is a Weyl or affine Weyl group, then $\mathrm{a}(w) = n(w)$.

Proposition (Green–X. 2018)

The result that $\mathrm{a}(w) = n(w)$ also holds for FC elements in *star-reducible* Coxeter groups.

It would be interesting to know whether the "$\mathrm{a} = n$" result actually holds for all Coxeter systems.

Star operations via heaps

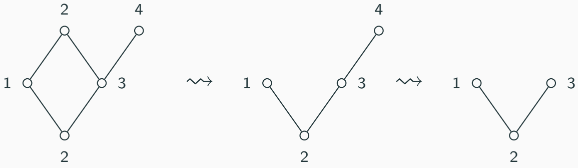
Let $I = \{s, t\}$ where $s, t \in S$ and $m(s, t) \geq 3$. Let $w \in W$.

- A *right lower star operation with respect to* I, denoted $-_*$, is sometimes defined on w; if so, then we have $w \sim_R w_*$.
- If w is FC, then we can detect when w_* is defined and visualize the operation in terms of heaps: w_* is defined iff one element $x \in I$ is maximal in $H(w)$ and the other element $y \in I \setminus \{x\}$ is maximal in $H(w) \setminus \{x\}$; in this case, the star operation removes x from w and yields $w_* = wx$.

Example: star operations via heaps

In the Coxeter group A_4, given by the graph

there is a unique way to start from 21324 and apply two lower right star operations in succession. Doing so results in 213, on which no more lower right star operations are applicable.



Stubs

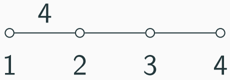
Define a (*left*) *stub* to be an element admitting no lower right star operations, and call a stub an a(2)-*stub* if it has a-value 2.

Then every a(2)-stub w is FC, and

- w must have exactly two right descents; the descents must commute;
- every "submaximal" generator in $H(w)$ must be adjacent to both the right descents of w in the Coxeter diagram.

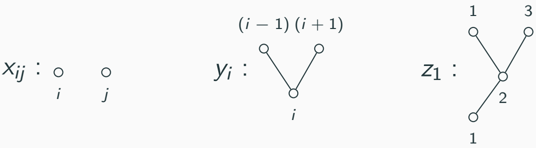
Example: a(2)-stubs of B_4

Consider the Coxeter system B_4 given by the following diagram.



The number of a(2)-stubs in B_4 is $\binom{4}{2} = 6$. The a(2)-stubs are:

- x_{13}, x_{14}, x_{24}, where $x_{ij} = ij$;
- y_2, y_3, where $y_i = i \cdot (i-1)(i+1)$;
- $z_1 := 1 \cdot 2 \cdot 13$.



Note that these stubs have pairwise distinct left descent sets.

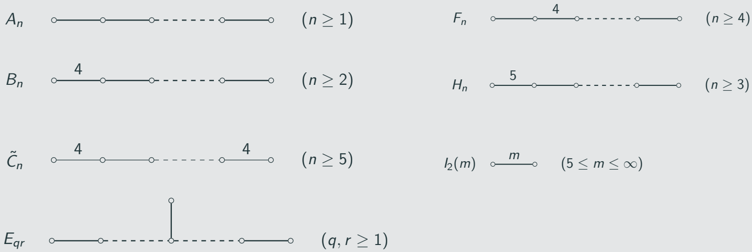
Selected Results

Classification of a(2)-finite Coxeter systems

We say a Coxeter system (W, S) is a(2)-*finite* if W_2 is finite.

Theorem (Green–X. 2018)

An irreducible Coxeter group is a(2)-finite if and only if its Coxeter diagram G is complete or one of the following graphs.



Moreover, $|W_2| = 0$ if and only if G is complete or of type $I_2(m)$.

Outline of proof

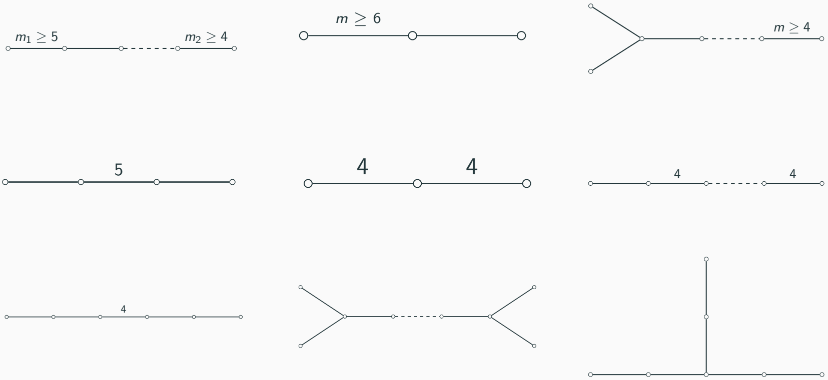
To prove the "if" direction,

- use Ernst's Temperley–Lieb diagrams for type $\tilde{C}$;
- use a separate argument involving heaps for E_{qr};
- use Stembridge's *FC-finite* classification for the other cases.

To prove the "only if" direction,

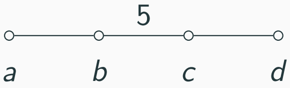
- find "forbidden subgraphs" that would violate a(2)-finiteness, by finding infinitly many "witnesses" of a-value 2 in each case;
- rule out forbidden subgraphs to obtain our graphs.

The forbidden configurations (for acyclic systems)

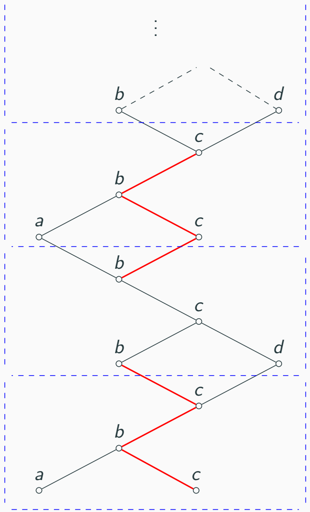


Example: witnesses for a difficult case

Forbidden configuration:



Witnesses:



Characterization of cells

Let (W, S) be an a(2)-finite Coxeter system. Let $\mathcal{S}$ be the set of a(2)-stubs in W. For each x, let

$$R_x = \{w \in W : w \text{ can be } \textit{right star reduced} \text{ to } x\}.$$

Theorem (Green–X. 2021)

We can find all KL cells in W_2 via $\mathcal{S}$:

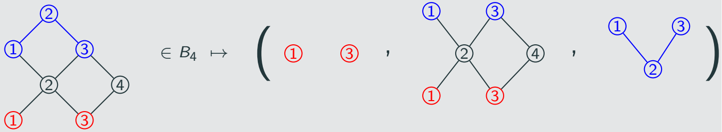
- The set $\mathcal{S}$ parameterizes the right cells of W_2: we have $W_2 = \sqcup_{x \in \mathcal{S}} R_x$, and for all $x \in W_2$ and $w \in W_2$, we have $w \sim_R x$ for $w \in W$ if and only if $w \in R_x$, if and only if w admits a *reduced factorization* $w = x \cdot z$ for some $z \in W$.
- We can also describe how the right cells $R_x (x \in \mathcal{S})$ coalesce into two-sided cells via a certain equivalence relation $\sim_*$ on $\mathcal{S}$.

Anatomy of a(2)-elements

Theorem (Green–X. 2021)

We can describe the structure of elements of W_2:

- There is a canonical bijection f from W_2 to the set of triples of the form (x, c, y) where $x, y \in \mathcal{S}$ and c is from a set $\mathrm{Cores}(x, y)$ of *cores compatible with* x *and* y.



- For fixed $x, y \in \mathcal{S}$, the map f restricts to a bijection from the "0-cell" $R_x \cap R_y^{-1}$ to $\mathrm{Cores}(x, y)$.

Cardinality of cells

Theorem (Green–X. 2021)

We are able to count all cells in W_2 as follows:

- reduce the counting problem to counting 0-cells $R_x \cap R_y^{-1}$ (and hence $\mathrm{Cores}(x, y)$) where $x, y \in \mathcal{S}$;
- further reduce the problem to counting a single 0-cell $\mathrm{Cores}(x, y)$ for two particular stubs x, y; in fact, we can use the equivalence relation $\sim_*$ on $\mathcal{S}$ to determine all 0-cells from any fixed 0-cell.
- count the above set $\mathrm{Cores}(x, y)$ using heaps.

Example: the group B_4

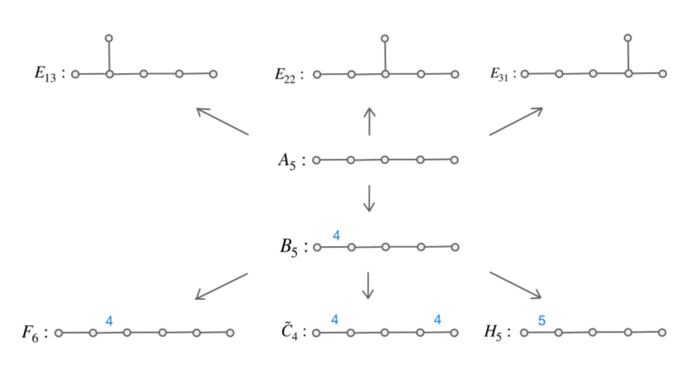
The following table records the 0-cells of a-value 2 in type B_4. The six elements of $\mathcal{S}$ label the row and columns of the table; the 0-cell $R_x \cap R_y^{-1}$ is given in Row x, Column y for all $x, y \in \mathcal{S}$.

	$1 \cdot 2 \cdot 13$	$2 \cdot 13$	13	14	24	$3 \cdot 24$
$1 \cdot 2 \cdot 13$	$\{121321, 1213241321\}$	$\{12132, 121324132\}$	$\{1213, 12132413\}$	$\{12134, 1213241\}$	$\{121324\}$	$\{1213243\}$
$2 \cdot 13$	$\{21321, 213241321\}$	$\{2132, 21324132\}$	$\{213, 2132413\}$	$\{2134, 213241\}$	$\{21324\}$	$\{213243\}$
13	$\{1321, 13241321\}$	$\{132, 1324132\}$	$\{13, 132413\}$	$\{134, 13241\}$	$\{1324\}$	$\{13243\}$
14	$\{41321, 1241321\}$	$\{4132, 124132\}$	$\{413, 12413\}$	$\{14, 1241\}$	$\{124\}$	$\{1243\}$
24	$\{241321\}$	$\{24132\}$	$\{2413\}$	$\{214\}$	$\{24, 2124\}$	$\{243, 21243\}$
$3 \cdot 24$	$\{3241321\}$	$\{324132\}$	$\{32413\}$	$\{3214\}$	$\{324, 32124\}$	$\{3243, 321243\}$

We can recover the table from any fixed entry. Adding cardinalities of 0-cells properly yields the cardinalities of all KL cells.

Proof ingredient: embeddings of Coxeter systems

It is possible to inductively compute the sets $\mathrm{Cores}(x, y) (x, y \in \mathcal{S})$ for all a(2)-finite Coxeter systems, by starting from type A and using certain embeddings:

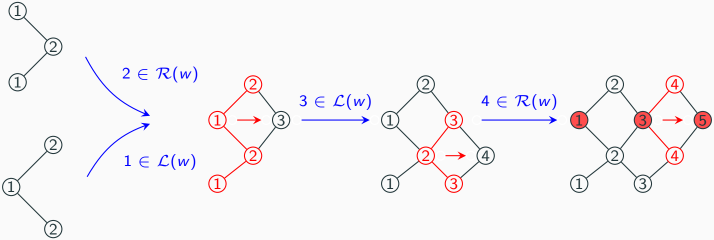


A network of weak embeddings reaches all a(2)-finite Coxeter types.

Proof ingredient: heap considerations

Heaps play a key role in our proofs. Here is an example argument:

Example: Let $I = R_{13} \cap R_{24}^{-1}$. Then $I = \{1324\}$ in type A_5, and we claim that $I = \{1324\}$ in type B_5. If not, take $w \in I \setminus \{1324\}$ in type B_5. Then $H(w)$ has a convex chain 121 or 212. This cannot happen since it would force $\mathrm{a}(w) \geq n(w) = 3$:



The vertical completions come from descent considerations; the horizontal completions are necessary to avoid FC-violating convex chains.

Stubs, cell modules, and a module of W

- Suppose (W, S) is of type "E_{pqr}" (three branches of sizes p, q, r, so that $E_{1qr} = E_{qr}$).
- Let V be the reflection representation of W. Let $M \subseteq V \otimes V$ be the kernel of the Coxeter bilinear form on V over a field of characteristic 0.
- Using cells modules associated to cells in W_2, we can show:

Theorem (Green–X., in preparation)

The W-module M has a basis on which the action of W is positive and can be combinatorially described via stubs. Moreover, the module M is reducible in types $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ and D_n, and it is irreducible otherwise.

Thank you!