Schur–Weyl duality for braid and twin groups via the Burau representation

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Schur–Weyl duality

The general setting is: a given (A, B)-bimodule M for algebras A, B.

Definition

M satisfies Schur–Weyl duality if the image of each action in $\mathrm{End}(M)$ *is equal to the centralizer for the other; that is:*

$$\mathrm{End}_A(M) = \mathrm{im}(rep(B)) \quad \textit{and} \quad \mathrm{End}_B(M) = \mathrm{im}(rep(A)).$$

Classically, A is a group algebra and M is a semisimple A-module.

- In that case, standard results in the theory of semisimple algebras can be applied, so the main task is to find a generic covering B of the centralizer $\mathrm{End}_A(M)$.
- Later it was found that some instances of Schur–Weyl duality continue to hold in non-semisimple situations.

This talk: Work over $\mathbb{C}$ in the classical (semisimple) setting.

Classical examples of Schur–Weyl duality

Let $E = \bigoplus_{i=1}^{n} \mathbb{C}e_i$ be an n-dimensional (complex) vector space with a fixed basis.

Identify $GL(E) \cong GL_n(\mathbb{C})$, $O(E) \cong O_n(\mathbb{C})$ via the basis. Write $\otimes = \otimes_{\mathbb{C}}$.

By [Schur 1927], [Brauer 1937], and [Jones 1994], the following commuting actions induce Schur–Weyl dualities:

$$\begin{aligned} GL_n \circlearrowleft E^{\otimes k} &\circlearrowright \mathcal{S}_k \\ O_n \circlearrowleft E^{\otimes k} &\circlearrowright \mathcal{B}_k(n) \\ W_n \circlearrowleft E^{\otimes k} &\circlearrowright \mathcal{P}_k(n) \end{aligned}$$

where each group on the left acts diagonally on the tensor power.

$W_n \subset O_n \subset GL_n$ all act on $E \cong \mathbb{C}^n$ by matrix multiplication. (So E is the standard permutation representation of W_n, the Weyl group in GL_n.)

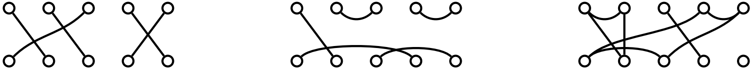
Diagram algebras on previous slide

$\mathcal{S}_k =$ symmetric group on k letters

$\mathcal{B}_k(n) =$ Brauer algebra on k strands

$\mathcal{P}_k(n) =$ partition algebra on k strands

Examples of diagrams in each algebra:



Multiplication is by a combinatorial 'stacking' procedure.

- The symmetric group $\mathcal{S}_k$ acts on $E^{\otimes k}$ via place-permutation in the tensor positions.
- $\mathcal{B}_k(n) \subset \mathcal{P}_k(n)$ act by a combinatorial rule (extends place-permutation).

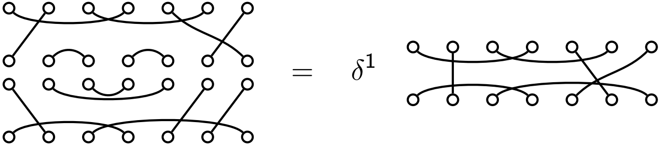
Multiplication in the partition algebra

To multiply d_1, d_2: stack d_1 above d_2, identify vertices in middle row, and count the number C of connected components in the middle. Then:

$$d_1 d_2 = \delta^C (d_1 \circ d_2)$$

where $d_1 \circ d_2$ is the diagram obtained by discarding all the middle vertices and middle connected components.

EXAMPLE:



Braid group and twin group

The symmetric group W_n on n letters is the group defined by generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

subject to the defining relations:

$$\sigma_i^2 = 1 \quad (1)$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1 \quad (2)$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1. \quad (3)$$

The braid group B_n and twin group TW_n are defined by omitting relations (1) and (2), respectively, in the above presentation. So they are covering groups of W_n.

Permutation representation

Recall that $E = \mathbb{C}e_1 \oplus \cdots \mathbb{C}e_n$, the natural permutation module for W_n, has a decomposition

$$E = F \oplus L$$

where $L = \mathbb{C}(e_1 + \cdots + e_n)$ and $F = \bigoplus_j \mathbb{C}(e_j - e_{j+1})$.

This is a decomposition into irreducible $\mathbb{C}[W_n]$-submodules.

Iwahori–Hecke algebra

Vaughan Jones (1987) suggested that to get interesting representations of braid groups we should look first at representations that factor through the Iwahori–Hecke algebra:

$$H_n(q_1, q_2) := \mathbb{C}[B_n]/((\sigma_i - q_1)(\sigma_i - q_2)).$$

Set $T_i :=$ image of σ_i in the quotient. Then $H_n(q_1, q_2)$ is the algebra generated by $T_1, \ldots, T_{n-1}$ subject to:

$$(T_i - q_1)(T_i - q_2) = 0 \tag{4}$$

$$T_i T_j T_i = T_j T_i T_j \text{ if } |i - j| = 1 \tag{5}$$

$$T_i T_j = T_j T_i \text{ if } |i - j| > 1. \tag{6}$$

Assume that $q_1 q_2 \neq 0$. With $q := -q_2/q_1$ we have isomorphisms:

$$H_n(q_1, q_2) \cong H_n(-1, q) \cong H_n(1, -q) \cong H_n(-q^{1/2}, q^{1/2}).$$

The Burau representation of the braid group

Define an action of B_n on $E = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ by:

$$\sigma_i \cdot e_j = \begin{cases} q_1 e_j & \text{if } j \neq i, i+1 \\ -q_2 e_{j-1} & \text{if } j = i+1 \\ (q_1 + q_2)e_j + q_1 e_{j+1} & \text{if } j = i. \end{cases}$$

This action makes E into a $\mathbb{C}[B_n]$-module. The action factors through $H_n(q_1, q_2)$, so it is also an $H_n(q_1, q_2)$-module.

- Equivalent (by rescaling) to the definition in [Burau 1935].
- Setting $(q_1, q_2) = (1, -1)$ we get the permutation rep of W_n.

Set $[n]_q := 1 + q + \cdots + q^{n-1}$. Define:

$$\ell_0 := e_1 + \cdots + e_n, \qquad f_i := qe_i - e_{i+1}.$$

Lemma (Kilmoyer?)

The vector ℓ_0 is invariant under the action, and $E = F \oplus L$ if and only if $[n]_q \neq 0$, as $H_n(q_1, q_2)$-modules, where $L := \mathbb{C}\ell_0$ and $F := \bigoplus_i \mathbb{C}f_i$.

Matrix form of the Burau rep

If we set Q equal to the 2×2 matrix

$$Q := \begin{bmatrix} q_1 + q_2 & -q_2 \\ q_1 & 0 \end{bmatrix}.$$

then σ_i (and T_i) acts on E via the $n \times n$ block diagonal matrix

$$\sigma_i \mapsto \begin{bmatrix} q_1 I_{i-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & q_1 I_{n-i-1} \end{bmatrix}.$$

Bilinear form

There is a nondegenerate symmetric bilinear form $\langle -, - \rangle$ on E such that

$$\langle e_i, e_j \rangle = \delta_{ij} q^{j-1}$$

where $q = -q_2/q_1$.

- The matrix of the form is $J = \mathrm{diag}(1, q, \ldots, q^{n-1})$.
- Notice that $\langle f_i, \ell_0 \rangle = 0$ for all $i = 1, \ldots, n-1$ (where $f_i := qe_i - e_{i+1}$).

Thus $F \subset L^\perp$ (the orthogonal complement with respect to the form).

Since $\dim L^\perp = n - 1$ by the standard theory of bilinear forms, it follows by dimension comparison that $F = L^\perp$.

- Set $O(E) := O_{\langle -,- \rangle}(E)$, orthogonal group.

The twin group representation

In $H_n(q_1, q_2)$ we define elements

$$S_i := \frac{1}{q_1 - q_2}(2T_i - (q_1 + q_2)).$$

These elements act on E as *reflections*, that is, orthogonal matrices with one eigenvalue equal to -1 and all other eigenvalues equal to 1.

So, as an operator on E, $S_i \in O(E) \cong O_n(\mathbb{C})$ and the map

$$\sigma_i \mapsto S_i$$

makes E into an orthogonal representation of TW_n.

- Still have $E = L \oplus F$ as $\mathbb{C}[TW_n]$-modules if and only if $[n]_q \neq 0$.
- S_i is in $O(L) \times O(F)$.

New instances of SWD

Theorem

Suppose that $q_1 q_2 \neq 0$ *and* $q = -q_2/q_1$ *is not a root of unity. Then the commuting actions* $\mathbb{C}[B_n] \circlearrowleft E^{\otimes k} \circlearrowright \mathcal{PP}_k([n]_q)$ *satisfy Schur–Weyl duality.*

Theorem

Assume that $[n]_q \neq 0$, $[n-2]_q^! \neq 0$, *and that*

$$q \neq \frac{-\lambda \pm \sqrt{-1-2\lambda}}{1+\lambda}$$

for any $\lambda = \cos(2k\pi/m)$ *with* $m \in \mathbb{Z}_{>0}$. *If* $\delta' \neq 0$, *the commuting actions*

$$\mathbb{C}[TW_n] \circlearrowleft E^{\otimes k} \circlearrowright \mathcal{PB}_k(n, \delta')$$

satisfy Schur–Weyl duality. The action of $\mathcal{PB}_k(n, \delta'))$ *is faithful if and only if* $n > k$.

Consequences

Corollary

Under the same hypotheses as Theorem 1, $E^{\otimes k}$ *has the bimodule decomposition*

$$E^{\otimes k} = \bigoplus_\lambda \Delta(\lambda) \otimes \mathcal{PP}^\lambda$$

where λ *varies over the set of all partitions of* j *with at most* $n-1$ *parts and* $0 \leq j \leq k$*. Here,* $\Delta(\lambda)$ *is a Weyl module for* GL_{n-1} *of highest weight* λ *and* $\mathcal{PP}^\lambda$ *is an irrep of* $\mathcal{PP}_k([n]_q)$*.*

Corollary

Under the same hypotheses as Theorem 2, $E^{\otimes k}$ *has the bimodule decomposition*

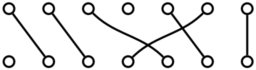
$$E^{\otimes k} = \bigoplus_\lambda \Delta(\lambda) \otimes \mathcal{PB}^\lambda$$

where λ *varies over the set of all partitions of* j *such that* $\lambda'_1 + \lambda'_2 \leq n-1$*, and* $0 \leq j \leq k$*. Here,* $\Delta(\lambda)$ *is a Weyl module for* O_{n-1} *of highest weight* λ *and* $\mathcal{PB}^\lambda$ *is an irrep of* $\mathcal{PB}_k(n, \delta')$*.*

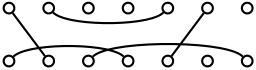
Remarks

1. The complicated hypotheses in the second theorem are always satisfied if q avoids the union of the positive real axis and the unit circle.

2. $\mathcal{PP}_k(\delta)$ is the *partial permutation algebra*, i.e., the subalgebra of $\mathcal{P}_k(\delta)$ spanned by k-diagrams with no horizontal edges and blocksizes of cardinality at most 2.



3. $\mathcal{PB}_k(\delta, \delta')$ is the two-parameter *partial Brauer algebra*, spanned by k-diagrams with blocksizes of cardinality at most 2. This algebra first appeared in [Martin and Mazorchuk, 2014], [Halverson and delMas, 2014].



Use δ, δ' to distinguish between loops, non-loops in the middle (an isolated vertex is consider a non-loop).

Proof sketch

Key step: show that the image of the action of B_n, TW_n on $E = L \oplus F$ is Zariski dense in

$$GL(L) \times GL(F), \qquad O(L) \times O(F)$$

respectively.

In the orthogonal case, this involves computations with Chebyshev polynomials which are motivated by the classical Rodrigueś rotation formula (1840) in Euclidean geometry.

Once density is established, we can plug into earlier proved versions of semisimple SWD to compute the centralizer.

Relation to some previous work

1. [Solomon, 2002] obtained a Schur–Weyl duality for the commuting actions

$$GL(F) \cong GL_{n-1}(\mathbb{C}) \circlearrowleft (\mathbb{C} \oplus F)^{\otimes k} \circlearrowright \mathbb{C}[\mathrm{Rook}_k]$$

   where Rook_k is the 'rook monoid' previously studied by W.D. Munn and many others.

2. [Martin and Mazorchuk, 2014] obtained a Schur–Weyl duality for the commuting actions

$$O(F) \cong O_{n-1}(\mathbb{C}) \circlearrowleft (\mathbb{C} \oplus F)^{\otimes k} \circlearrowright \mathcal{PB}_k(n, 1).$$

3. The inclusion of the trivial summand before forming tensor powers in the above results is somewhat artificial. Our viewpoint provides a more natural context in which to interpret these statements.

References

1. D and Giaquinto, *Schur–Weyl duality for tensor powers of the Burau representation*, Research in Math Sci (2021).
2. D and Giaquinto, *Schur–Weyl duality for twin groups*, arXiv (2021).