Sylow Branching Coefficients & a conjecture of Malle & Navarro

- G finite group, $\mathrm{Irr}(G) = \{$ irreducible characters of $G\}$
- $\mathbb{1}_G$: trivial char. of G.
- For p prime, $\mathrm{Syl}_p(G) = \{$ Sylow p-subgps of $G\}$

① Motivation

- Char. theory of Sylow subgps
  $\rightsquigarrow$ Themes: - to understand the relationship between G & its subgps
  - what do irred. chars tell us about group structure?

- e.g. $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$
  $|G/G'| = \#$ linear characters of G (deg. 1)

- Itô-Michler Theorem: G finite gp, p prime, $P \in \mathrm{Syl}_p(G)$.
  Then $P \trianglelefteq G$ and P is abelian $\iff p \nmid \chi(1)$ $\forall \chi \in \mathrm{Irr}(G)$

- Qs:
  P abelian $\iff$ (character properties?)
  $P \trianglelefteq G$ $\iff$ (character properties?)

- Malle & Navarro 2021: G finite gp, p prime, $P \in \mathrm{Syl}_p(G)$
  Then P abelian $\iff$ $p \nmid \chi(1)$ $\forall$ $\underbrace{\chi \in \mathrm{Irr}(B_0(G))}_{\text{subset of } \mathrm{Irr}(G) \text{ in principal } p\text{-block of } G}$

[P satisfies « property » $\iff$ $p \nmid \chi(1)$ $\forall$ $\chi \in$ « subset of $\mathrm{Irr}(G)$ »]

- Malle & Navarro 2012: G finite gp, p prime, $P \in \mathrm{Syl}_p(G)$.
  Then $P \trianglelefteq G$ $\iff$ $p \nmid \chi(1)$ $\forall$ irred. constituents χ of $\mathbb{1}_P\big\uparrow^G$.

- Note $P \leq N_G(P) \leq G$

  equality determined by [MN2012]

- Navarro, Tiep, Vallejo 2014: G finite gp, p odd prime, $P \in \mathrm{Syl}_p(G)$

  then $P = N_G(P) \iff \mathbb{1}_G$ is the only irred. constituent of $\mathbb{1}_P \uparrow^G$ of degree coprime to p

  [false when $p=2$: e.g. $G = S_5$]

$\rightsquigarrow$ " normality of P is controlled by $\mathbb{1}_P \uparrow^G$ "

- Q. what are the irred. constituents of $\mathbb{1}_P \uparrow^G$?

- For $\chi \in \mathrm{Irr}(G)$, $P \in \mathrm{Syl}_p(G)$, $\phi \in \mathrm{Irr}(P)$,

  Sylow Branching Coefficient (SBC) $Z^\chi_\phi := \langle \chi, \phi \uparrow_P^G \rangle$

  $= \langle \chi \downarrow_P, \phi \rangle$

$\rightsquigarrow$ [MN2012]: $P \trianglelefteq G \iff p \nmid \chi(1) \quad \forall \underbrace{\chi \in \mathrm{Irr}(G) \text{ s.t. } Z^\chi_{\mathbb{1}_P} > 0}_{\text{subset of } \mathrm{Irr}(G) \text{ relating to positivity of SBCs}}$

- Conjecture (Malle & Navarro 2012): G finite gp, p prime, $P \in \mathrm{Syl}_p(G)$

  then $P \trianglelefteq G \iff p \nmid \chi(1) \quad \forall \underbrace{\chi \in \mathrm{Irr}(G) \text{ s.t. } p \nmid Z^\chi_{\mathbb{1}_P}}_{\text{subset of } \mathrm{Irr}(G) \text{ relating to divisibility of SBCs.}}$

($\Rightarrow$) holds since $p \nmid Z^\chi_{\mathbb{1}_P} \Rightarrow Z^\chi_{\mathbb{1}_P} > 0$.

($\Leftarrow$) : joint work with E. Giannelli, J. Long, C. Vallejo

Idea : - want to show if $P \ntrianglelefteq G$, then $\exists\ \chi \in \mathrm{Irr}(G)$

s.t. $p \nmid z^{\chi}_{\mathbb{1}_P}$ but $p \mid \chi(1)$.

- reduce to simple gps: want to show every non-abelian finite simple gp S has some $\chi \in \mathrm{Irr}(S)$ s.t. $p \nmid z^{\chi}_{\mathbb{1}_P}$ but $p \mid \chi(1)$.

- Difficult case: $S = A_n$

$\rightsquigarrow$ Study A_n via our focus S_n: the symmetric gp

② Investigating SBCs for S_n

- Translating to S_n: suffices to find $\chi \in \mathrm{Irr}(S_n)$ s.t.

$$\underbrace{p \nmid z^{\chi}_{\mathbb{1}_P}}_{\text{SBC condition}} \quad \text{and} \quad \underbrace{p \mid \chi(1)}_{\text{degree condition}}$$

- Calculating individual SBCs in general: hard

e.g. [Giannelli, L. 2018]: computed when $z^{\chi}_{\mathbb{1}_P} > 0$ for all n & odd primes p.

but $p=2$: still open.

- But: linear combinations could be nicer.

$\rightsquigarrow$ if $p \nmid \sum_{\text{some } \chi} z^{\chi}_{\mathbb{1}_P}$ then $\exists\, \chi^*$ s.t. $p \nmid z^{\chi^*}_{\mathbb{1}_P}$

- Example: ρ_G regular character of G, $\rho_G(g) = \begin{cases} |G| & \text{if } g=1 \\ 0 & \text{o/w.} \end{cases}$

Fact: $\rho_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \cdot \chi$

$\rightsquigarrow$ SBC for $\rho_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \cdot z^{\chi}_{\mathbb{1}_P}$

$= \langle \rho_G, \mathbb{1}_P\!\uparrow^G \rangle$

$$= \langle \rho_G\downarrow_P, \mathbb{1}_P \rangle = \left\langle \frac{|G|}{|P|}\cdot \rho_P , \mathbb{1}_P \right\rangle$$

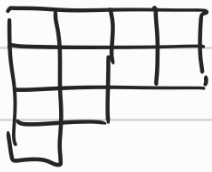
$$= \frac{|G|}{|P|} \text{ , coprime to } p.$$

• $\boxed{S_n}$ $\quad \mathrm{Irr}(S_n) \xleftrightarrow{1-1} \{\text{partitions of } n\}$

$$\chi^\lambda \longleftrightarrow \lambda$$

- Partition of n : e.g. $\lambda = (4,4,2,1) \vdash 11$

Young diagram $Y(\lambda)$:



- degree $\chi^\lambda(1)$ = # of ways of filling in $1,2,3,\dots,n$ into the boxes of $Y(\lambda)$ s.t. rows increase left to right, columns increase top to bottom.

$\rightsquigarrow$ $\chi^\lambda(1)$ = [start with ϕ . fix top row, leftmost col.] # of ways of adding on <u>one</u> box at a time s.t. intermediate shapes are genuine Young diagrams & final shape is $Y(\lambda)$

e.g. $\lambda = (2,2)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\checkmark \qquad \checkmark \qquad \times \qquad \times \qquad \text{etc.}$$

$$\chi^{(2,2)}(1) = 2$$

• Reg. char.

$$\underline{\rho_{S_n}} = \sum_{\chi^\lambda \in \mathrm{Irr}(S_n)} \underline{\chi^\lambda(1)} \cdot \chi^\lambda$$

scalar multiple of indicator class fn on cycle type: $(1,1,1,\dots,1)$

# of ways of adding on this many boxes at each step: $(1,1,1,\dots,1)$

- We construct, for each general cycle type $\sigma = (k_1, k_2, \dots, k_t)$

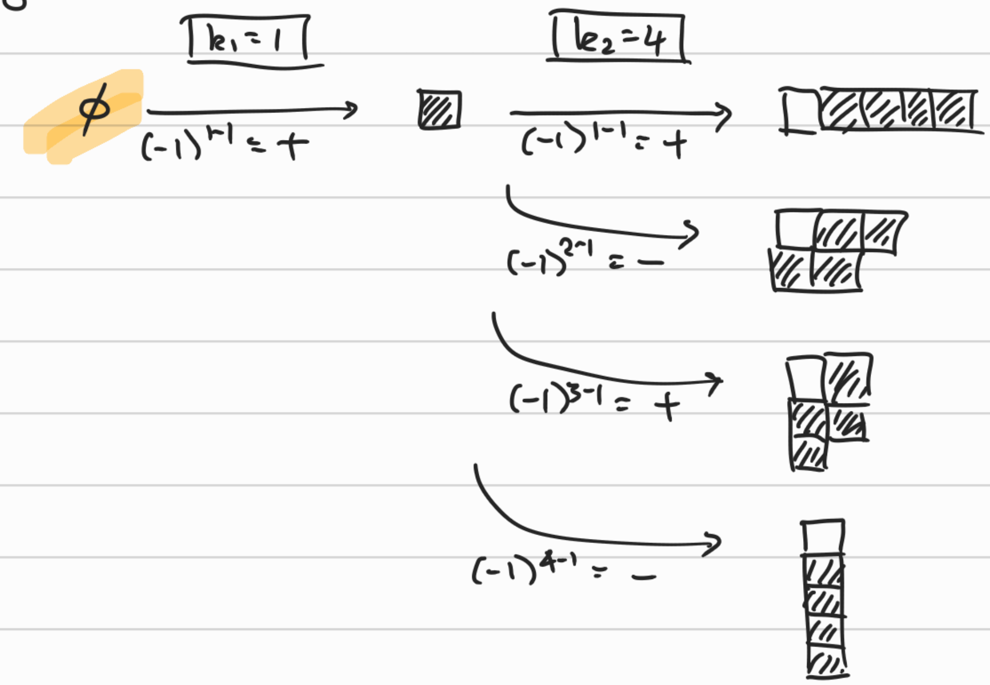
$$\underbrace{\rho_\sigma}_{\uparrow} := \sum_{\chi^\lambda \in \mathrm{Irr}(S_n)} \underbrace{c^\lambda(\sigma)}_{\uparrow} \cdot \chi^\lambda$$

[GLLV]: scalar multiple of ind. class fn. of cycle type σ = $(k_1, k_2, \dots, k_t)$

# of ways of adding on strips (hooks) of this many boxes at each step: $(k_1, k_2, \dots, k_t)$, with certain signs (leg length)

⌜ choose σ to get p ✗ SBC ⌟

eg. $n=5$, $\sigma \in S_5$ cycle type $(4,1)$, $k_1 = 1$, $k_2 = 4$



hooks: connected shapes w/ no 2×2

$$\therefore \rho_\sigma = \chi^{(5)} - \chi^{(3,2)} + \chi^{(2,2,1)} - \chi^{(1^5)}$$

$\rightsquigarrow$ extend construction by starting at any λ rather than ϕ

⌜ choose λ to control degree ⌟

$P_n \in \mathrm{Syl}_p(S_n)$; P_{p^k} $\qquad \mathrm{Lin}(P_{p^k}) \longleftrightarrow \{0, 1, \dots, p-1\}^k$

$(C_p)^k$

$M^\lambda : \quad \mathbb{1} \uparrow_{S_\lambda}^{S_n}$