# Intermediate Symplectic Characters and Enumeration of Shifted Plane Partitions

Soichi OKADA (Nagoya University)

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## Plan:

- Intermediate symplectic characters
- Determinant formulas
- Application to enumeration of shifted plane partitions

## References


- R. A. Proctor,
  Odd symplectic groups,
  Invent. Math. **92** (1988), 307–332.
- S. Okada,
  Intermediate symplectic characters and shifted plane partitions of shifted double staircase shape,
  Comb. Theory **1** (2021), No. 10.

# Intermediate Symplectic Characters

## Schur functions (irreducible characters of $\mathrm{GL}_n$)

Let $n$ be a positive integer, and $\lambda$ a partition of length $\leq n$. An $n$-semistandard tableau of shape $\lambda$ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from $1, 2, \ldots, n$ such that

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;

Then the Schur function $s_\lambda(x_1, \ldots, x_n)$ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \boldsymbol{x}^T, \quad \boldsymbol{x}^T = \prod_{i=1}^n x_i^{\#(i\text{'s in } T)},$$

where $T$ runs over all $n$-semistandard tableaux of shape $\lambda$.

$T =$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 4 |
| 2 | 3 | 3 | | |
| 4 | 4 | | | |

$, \quad \boldsymbol{x}^T = x_1^2 x_2^3 x_3^2 x_4^3$

## Symplectic characters (irreducible characters of $\mathrm{Sp}_{2n}$)

An $n$-symplectic tableau of shape $\lambda$ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$ such that

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;
- the entries in the $i$th row are greater than or equal to $i$.

Then the symplectic character $\mathrm{sp}_\lambda(x_1, \dots, x_n)$ is defined by

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \sum_T \boldsymbol{x}^T, \quad \boldsymbol{x}^T = \prod_{i=1}^n x_i^{\#(i\text{'s in } T) - \#(\overline{i}\text{'s in } T)},$$

where $T$ runs over all $n$-symplectic tableaux of shape $\lambda$.

$T =$

| | | | | |
|---|---|---|---|---|
| $1$ | $1$ | $\overline{1}$ | $\overline{2}$ | $4$ |
| $2$ | $\overline{2}$ | $3$ | | |
| $\overline{4}$ | $5$ | | | |

, $\quad \boldsymbol{x}^T = x_1 x_2^{-1} x_3 x_5$

## Intermediate symplectic tableaux

Let $n$ and $k$ be two integers such that $n > 0$ and $0 \leq k \leq n$, and $\lambda$ a partition of length $\leq n$. A $(k, n-k)$-symplectic tableau of shape $\lambda$ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < k < \overline{k} < k+1 < k+2 < \cdots < n\}$$

satisfying the following three conditions:

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;
- the entries in the $i$th row are greater than or equal to $i$.

For example,

$$T = \begin{array}{|c|c|c|c|c|} \hline \overline{1} & 2 & 2 & 3 & 4 \\ \hline 2 & \overline{2} & 5 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array}$$

is a $(2, 3)$-symplectic tableau of shape $(5, 3, 2)$.

## Intermediate symplectic tableaux

Let $n$ and $k$ be two integers such that $n > 0$ and $0 \leq k \leq n$, and $\lambda$ a partition of length $\leq n$. A $(k, n-k)$-symplectic tableau of shape $\lambda$ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < k < \overline{k} < k+1 < k+2 < \cdots < n\}$$

satisfying the following three conditions:

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;
- the entries in the $i$th row are greater than or equal to $i$.

If $k = 0$ or $k = n$, then

$$(0, n)\text{-symplectic tableaux} = n\text{-semistandard tableaux},$$
$$(n, 0)\text{-symplectic tableaux} = n\text{-symplectic tableaux}.$$

## Intermediate symplectic characters

Given a partition $\lambda$ of length $\leq n$, we define the $(k, n-k)$-symplectic character $\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n)$ by

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n) = \sum_T \boldsymbol{x}^T,$$

where $T$ runs over all $(k, n-k)$-symplectic tableaux of shape $\lambda$, and

$$\boldsymbol{x}^T = \prod_{i=1}^{k} x_i^{\#(i\text{'s in } T) - \#(\bar{i}\text{'s in } T)} \prod_{i=k+1}^{n} x_i^{\#(i\text{'s in } T)}.$$

If $k = 0$ or $k = n$, then

$$\mathrm{sp}_\lambda^{(0,n)}(\boldsymbol{x}) = s_\lambda(\boldsymbol{x}) : \text{Schur function},$$
$$\mathrm{sp}_\lambda^{(n,0)}(\boldsymbol{x}) = \mathrm{sp}_\lambda(\boldsymbol{x}) : \text{symplectic character}.$$

## Intermediate symplectic groups

Let $V = \mathbb{C}^{n+k}$ be the $(n+k)$-dimensional complex vector space with basis $e_1, e_{\overline{1}}, \ldots, e_k, e_{\overline{k}}, e_{k+1}, \ldots, e_n$. Let $\langle\ ,\ \rangle$ be the skew-symmetric bilinear form (not necessarily non-degenerate) on $V$ defined by

$$\langle e_\alpha, e_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = i \text{ and } \beta = \overline{i} \text{ for } 1 \le i \le k, \\ -1 & \text{if } \alpha = \overline{i} \text{ and } \beta = i \text{ for } 1 \le i \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Then the intermediate symplectic group $\mathbf{Sp}_{2k,n-k}$ is defined by

$$\mathbf{Sp}_{2k,n-k} = \{g \in \mathbf{GL}(V) : \langle gv, gw \rangle = \langle v, w \rangle\ (v, w \in V)\}.$$

If $k = 0$ or $k = n$, then

$$\mathbf{Sp}_{0,n} \cong \mathbf{GL}_n, \quad \mathbf{Sp}_{2n,0} \cong \mathbf{Sp}_{2n}.$$

If $k = n-1$, then the group $\mathbf{Sp}_{2k,1}$ is called the odd symplectic group.

## Representations of intermediate symplectic groups

Recall that $\mathbf{Sp}_{2k,n-k} \subset \mathbf{GL}(V)$ with $V = \mathbb{C}^{n+k}$.

**Theorem** (Proctor) Let $\lambda$ be a partition of $d$ with length $\leq n$. Let

$V^\lambda =$ an irreducible $\mathbf{GL}_{n+k}$-submodule of $V^{\otimes d}$ corresp. to $\lambda$,

$V_0^\lambda =$ the trace-free subspace of $V^\lambda$.

Then

- $V_0^\lambda$ is an indecomposable $\mathbf{Sp}_{2k,n-k}$-module.
- $V_0^\lambda$ has a weight basis indexed by $(k, n-k)$-symplectic tableaux of shape $\lambda$.

Hence

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \ldots, x_k | x_{k+1}, \ldots, x_n) = \text{the character of } V_0^\lambda.$$

## Determinant Formulas
## — Jacobi–Trudi and bialternant formulas —

## Jacobi–Trudi formulas for $s_\lambda$ and $\mathrm{sp}_\lambda$

For a partition of length $\le n$, we have

$$s_\lambda(x_1, \dots, x_n) = \det\left(h_{\lambda_i - i + j}(x_1, \dots, x_n)\right)_{1 \le i, j \le n}$$
$$= \det\left(h_{\lambda_i - i + j}(x_j, \dots, x_n)\right)_{1 \le i, j \le n},$$

and

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \frac{1}{2} \det\begin{pmatrix} h_{\lambda_i - i + j}(x_1^{\pm 1}, \dots, x_n^{\pm 1}) \\ \quad + h_{\lambda_i - i - j + 2}(x_1^{\pm 1}, \dots, x_n^{\pm 1}) \end{pmatrix}_{1 \le i, j \le n}$$
$$= \det\left(h_{\lambda_i - i + j}(x_j^{\pm 1}, \dots, x_n^{\pm 1})\right)_{1 \le i, j \le n},$$

where $h_r(x_1, \dots, x_n)$ is the $r$th complete symmetric polynomial in $x_1$, $\dots, x_n$ and

$$h_r(x_1^{\pm 1}, \dots, x_n^{\pm 1}) = h_r(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}).$$

## Flagged Jacobi–Trudi formula for $\mathrm{sp}^{(k,n-k)}$ (1/2)

**Proposition** For a partition $\lambda$ of length $\le n$, we have

$$\mathrm{sp}^{(k,n-k)}_{\lambda}(\boldsymbol{x}) = \det \left( \begin{cases} h_{\lambda_i - i + j}(x_j^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } 1 \le j \le k \\ h_{\lambda_i - i + j}(x_j, \dots, x_n) & \text{if } k+1 \le j \le n \end{cases} \right),$$

where

$$h_r(x_j^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) = h_r(x_j, x_j^{-1}, \dots, x_k, x_k^{-1}, x_{k+1}, \dots, x_n).$$

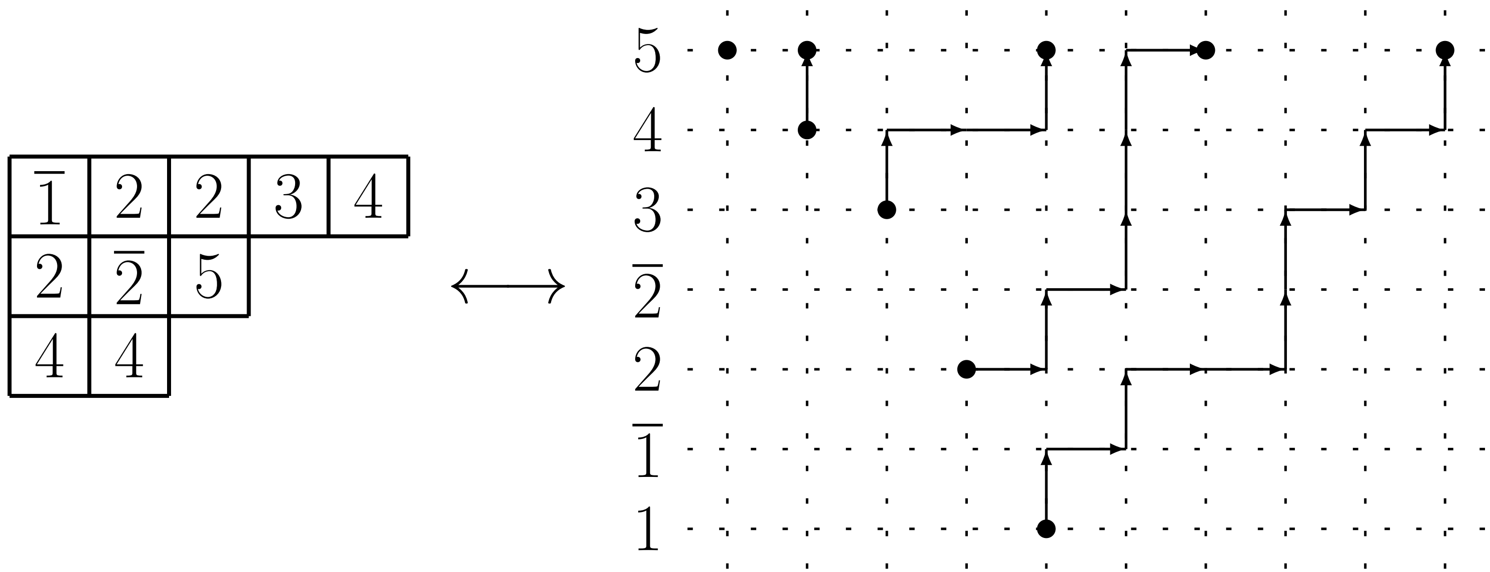
This formula reduces to the flagged Jacobi–Trudi formulas for Schur functions ($k = 0$) and symplectic Schur functions ($k = n$).

## Flagged Jacobi–Trudi formula for $\mathrm{sp}^{(k,n-k)}$ (2/2)

**Proposition** For a partition $\lambda$ of length $\leq n$, we have

$$\mathrm{sp}_{\lambda}^{(k,n-k)}(\boldsymbol{x}) = \det \left( \begin{cases} h_{\lambda_i - i + j}(x_j^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } 1 \leq j \leq k \\ h_{\lambda_i - i + j}(x_j, \dots, x_n) & \text{if } k+1 \leq j \leq n \end{cases} \right).$$

Proof idea We use an lattice path interpretation of intermediate symplectic tableaux and appeal to the Lindström–Gessel–Viennot Lemma.



## Jacobi–Trudi formula for $\mathrm{sp}^{(k,n-k)}$

By performing column operations on the flagged Jacobi–Trudi determinant, we obtain

**Proposition** For a partition $\lambda$ of length $\leq n$, we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(\boldsymbol{x}) = \det H_\lambda^{(k,n-k)},$$

where $H_\lambda^{(k,n-k)}$ is the $n \times n$ matrix with $(i,j)$ entry given by

$$\begin{cases} h_{(\lambda_i-i+1)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } j = 1, \\ h_{(\lambda_i-i+j)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \\ \quad + h_{(\lambda_i-i-j+2)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } 2 \leq j \leq k, \\ h_{\lambda_i-i+j}(x_{k+1}, \dots, x_n) & \text{if } k+1 \leq j \leq n. \end{cases}$$

This formula reduces to the Jacobi–Trudi formulas for Schur functions ($k = 0$) and symplectic Schur functions ($k = n$).

## Bialternant formulas for $s_\lambda$ and $\mathrm{sp}_\lambda$

The Schur functions and the symplectic Schur functions are expressed as ratios of two determinants:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det \left( x_j^{\lambda_i + n - i} \right)_{1 \le i, j \le n}}{\det \left( x_j^{n-i} \right)_{1 \le i, j \le n}},$$

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \frac{\det \left( x_j^{\lambda_i + n - i + 1} - x_j^{-(\lambda_i + n - i + 1)} \right)_{1 \le i, j \le n}}{\det \left( x_j^{n-i+1} - x_j^{-(n-i+1)} \right)_{1 \le i, j \le n}},$$

and

$$\det \left( x_j^{n-i} \right)_{1 \le i, j \le n} = \prod_{1 \le i < j \le n} (x_i - x_j),$$

$$\det \left( x_j^{n-i+1} - x_j^{-(n-i+1)} \right)_{1 \le i, j \le n} = \prod_{i=1}^{n} (x_i - x_i^{-1}) \prod_{1 \le i < j \le n} (x_i^{1/2} x_j^{1/2} - x_i^{-1/2} x_j^{-1/2})(x_i^{1/2} x_j^{-1/2} - x_i^{-1/2} x_j^{1/2}).$$

## Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (1/3)

**Theorem** Given a partition $\lambda$ of length $\leq n$, we define $A_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with $(i,j)$ entry given by

$$\begin{cases} h_{\lambda_i+k-i+1}(x_j, x_{k+1}, \dots, x_n) - h_{\lambda_i+k-i+1}(x_j^{-1}, x_{k+1}, \dots, x_n) & \text{if } 1 \leq j \leq k, \\ x_j^{\lambda_i+n-i} & \text{if } k+1 \leq j \leq n, \end{cases}$$

Then we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n) = \frac{\det A_\lambda^{(k,n-k)}}{\det A_\emptyset^{(k,n-k)}}.$$

This formula reduces to the bialternant formulas for Schur functions $(k=0)$ and symplectic Schur functions $(k=n)$.

## Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (1/3)

**Theorem** Given a partition $\lambda$ of length $\leq n$, we define $A_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with $(i,j)$ entry given by

$$\begin{cases} h_{\lambda_i+k-i+1}(x_j, x_{k+1}, \dots, x_n) - h_{\lambda_i+k-i+1}(x_j^{-1}, x_{k+1}, \dots, x_n) & \text{if } 1 \leq j \leq k, \\ x_j^{\lambda_i+n-i} & \text{if } k+1 \leq j \leq n, \end{cases}$$

Then we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n) = \frac{\det A_\lambda^{(k,n-k)}}{\det A_\emptyset^{(k,n-k)}}.$$

Remark The denominator factors as

$$\det A_\emptyset^{(k,n-k)} = \prod_{i=1}^{k} (x_i - x_i^{-1}) \prod_{1 \leq i < j \leq k} \left(x_i^{1/2} x_j^{1/2} - x_i^{-1/2} x_j^{-1/2}\right)\left(x_i^{1/2} x_j^{-1/2} - x_i^{-1/2} x_j^{1/2}\right) \times \prod_{k+1 \leq i < j \leq n} (x_i - x_j).$$

## Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (2/3)

Sketch of Proof  We can find an $n \times n$ matrix $M$ such that

$$A_\lambda^{(k,n-k)} = H_\lambda^{(k,n-k)} M$$

for any partitions $\lambda$ of length $\leq n$. Since $H_\emptyset^{(k,n-k)}$ is a upper-triangular matrix with diagonal entries $1$, we have

$$\det A_\emptyset^{(k,n-k)} = \det H_\emptyset^{(k,n-k)} \cdot \det M = \det M.$$

Hence we obtain

$$\det A_\lambda^{(k,n-k)} = \det H_\lambda^{(k,n-k)} \cdot \det M = \det H_\lambda^{(k,n-k)} \cdot \det A_\emptyset^{(k,n-k)},$$

and

$$\mathrm{sp}_\lambda^{(k,n-k)}(\boldsymbol{x}) = \det H_\lambda^{(k,n-k)} = \frac{\det A_\lambda^{(k,n-k)}}{\det A_\emptyset^{(k,n-k)}}.$$

## Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (3/3)

**Corollary** Given a partition $\lambda$ of length $\leq n$, we define $\overline{A}_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with $(i,j)$ entry given by

$$\overline{a}_{i,j} = \begin{cases} \dfrac{x_j^{\lambda_i+k-i+1}}{\prod_{l=k+1}^{n}(1 - x_j^{-1} x_l)} - \dfrac{x_j^{-(\lambda_i+k-i+1)}}{\prod_{l=k+1}^{n}(1 - x_j x_l)} & \text{if } 1 \leq j \leq k, \\ x_j^{\lambda_i+n-i} & \text{if } k+1 \leq j \leq n. \end{cases}$$

Then we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n) = \frac{\det \overline{A}_\lambda^{(k,n-k)}}{\det \overline{A}_\emptyset^{(k,n-k)}}.$$

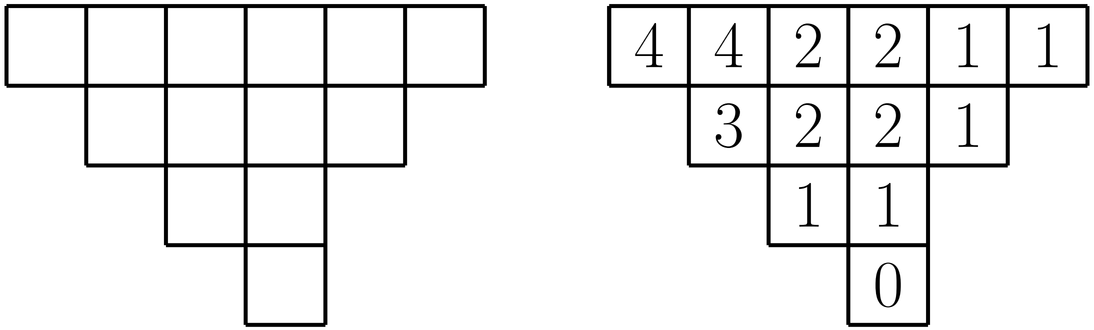
This formula also reduces to the bialternant formulas for Schur functions ($k = 0$) and symplectic Schur functions ($k = n$).

# Application to Enumeration of Shifted Plane Partitions

## Shifted plane partitions

Given a strict partition $\mu$, a shifted plane partition of shape $\mu$ is a filling of the shifted Young diagram $S(\mu)$ with nonnegative integers where the entries are weakly decreasing along rows and down columns.

Example




are the shifted diagram $S(6, 4, 2, 1)$ and a shifted plane partition of shape $(6, 4, 2, 1)$ respectively.

We put

$$\mathcal{A}^m(S(\mu)) = \{\text{shifted plane partitions of shape } \mu \text{ with entries} \le m\}.$$

## Shifted plane partitions of double staircase shape

Let $\delta_r = (r, r-1, \ldots, 2, 1)$ be the staircase partition of length $r$.

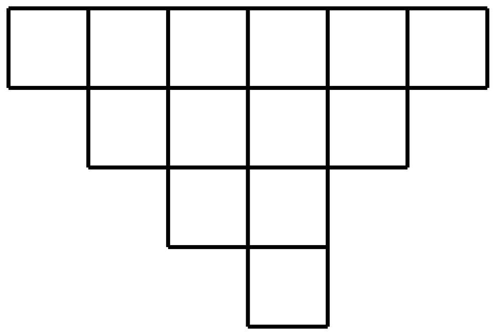
**Theorem** (Hopkins' Conjecture) If $0 \le k \le n$, then the number of shifted plane partitions of shape

$$\delta_n + \delta_k = (n+k, n+k-2, \ldots, n-k+2, n-k, n-k-1, \ldots, 2, 1)$$

with entries bounded by $m$ is equal to

$$\#\mathcal{A}^m(S(\delta_n + \delta_k)) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

$$S(\delta_4 + \delta_2) =$$



Remark Hopkins and Lai prove this theorem by counting lozenge tilings of a certain region in the triangular lattice.

## Shifted plane partitions of double staircase shape

Let $\delta_r = (r, r-1, \ldots, 2, 1)$ be the staircase partition of length $r$.

**Theorem** (Hopkins' Conjecture) If $0 \le k \le n$, then the number of shifted plane partitions of shape

$$\delta_n + \delta_k = (n+k, n+k-2, \ldots, n-k+2, n-k, n-k-1, \ldots, 2, 1)$$

with entries bounded by $m$ is equal to

$$\# \mathcal{A}^m(S(\delta_n + \delta_k)) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

Remark The case $k = 0$ is (the $q = 1$ case of) the MacMahon Conjecture on symmetric plane partitions, which was proved by Andrews and Macdonald independently in 1970s. The cases $k = n-1$ and $k$ were obtained by Proctor in 1990s.

## Bijection

For a shifted plane partition $\sigma$, we define the profile of $\sigma$ to be the partition $(\sigma_{1,1}, \sigma_{2,2}, \dots)$.

**Lemma** For a partition $\lambda$, there exists a bijection between

- shifted plane partitions of shape $\delta_n + \delta_k$ with profile $\lambda$, and
- $(k, n-k)$-symplectic tableaux of shape $\lambda$.

Example If $n = 4$ and $k = 2$, then

| | | | | | |
|---|---|---|---|---|---|
| 4 | 4 | 2 | 2 | 1 | 1 |
| | 3 | 2 | 2 | 1 | |
| | | 1 | 1 | | |
| | | | 0 | | |

$\xrightarrow{(1)}$

| | | | |
|---|---|---|---|
| 6 | 4 | 4 | 2 |
| 4 | 3 | 1 | |
| 2 | | | |

$\xrightarrow{(2)}$

| | | | |
|---|---|---|---|
| 1 | 2 | 2 | 3 |
| 2 | $\overline{2}$ | 4 | |
| 3 | | | |

(1) conjugate each row;
(2) replace $1, 2, 3, 4, 5, 6$ with $4, 3, \overline{2}, 2, \overline{1}, 1$ respectively.

## Bijection

For a shifted plane partition $\sigma$, we define the profile of $\sigma$ to be the partition $(\sigma_{1,1}, \sigma_{2,2}, \dots)$.

**Lemma** For a partition $\lambda$, there exists a bijection between

- shifted plane partitions of shape $\delta_n + \delta_k$ with profile $\lambda$, and
- $(k, n-k)$-symplectic tableaux of shape $\lambda$.

Hence the proof of Hopkins' Conjecture is reduced to showing

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_\lambda^{(k,n-k)}(1, \dots, 1|1, \dots, 1) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j},$$

where $\lambda$ runs over all partitions whose Young diagrams are contained in the $m \times n$ rectangle.

## Character identity

**Theorem** Let $0 \le k \le n$. For a nonnegative integer $m$, we have

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_{\lambda}^{(k,n-k)}(x_1,\dots,x_k | x_{k+1},\dots,x_n)$$
$$= \mathrm{o}^{B}_{((m/2)^n)}(x_1,\dots,x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1,\dots,x_k) \cdot (x_{k+1}\cdots x_n)^{m/2},$$

where $\mathrm{o}^B_\nu$ and $\mathrm{sp}_\nu$ denote the odd orthogonal and symplectic character respectively:

$$\mathrm{o}^B_\mu(x_1,\dots,x_n) = \frac{\det\left(x_i^{\mu_i+n-i+1/2} - x_i^{-(\mu_i+n-i+1/2)}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-i+1/2} - x_i^{-(n-i+1/2)}\right)_{1\le i,j\le n}},$$

$$\mathrm{sp}_\nu(x_1,\dots,x_k) = \frac{\det\left(x_i^{\nu_i+k-i+1} - x_i^{-(\nu_i+k-i+1)}\right)_{1\le i,j\le k}}{\det\left(x_i^{k-i+1} - x_i^{-(k-i+1)}\right)_{1\le i,j\le k}}.$$

## Character identity

**Theorem** Let $0 \le k \le n$. For a nonnegative integer $m$, we have

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$
$$= \mathrm{o}^B_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2},$$

If $k = 0$, then we have

$$\sum_{\lambda \subset (m^n)} s_\lambda(x_1, \dots, x_n) = \mathrm{o}^B_{((m/2)^n)}(x_1, \dots, x_n) \cdot (x_1 \cdots x_n)^{m/2},$$

which Macdonald used to prove the MacMahon and Bender–Knuth conjectures on symmetric plane partitions. If $k = n$, then we have

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_\lambda(x_1, \dots, x_n) = s_{(m^n)}(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1),$$

which was obtained by Proctor.

## Proof of the character identity (1/3)

We apply the minor summation formula of Ishikawa and Wakayama.

**Lemma** Let $n$ be a positive even integer. Let $X = (x_{i,j})_{1 \le i,\, j \le N}$ be an $N \times N$ skew-symmetric matrix and $Y = (y_{i,j})_{1 \le i \le n,\, 1 \le j \le N}$ be an $n \times N$ matrix. Then we have

$$\sum_{I} \operatorname{Pf} X(I) \cdot \det Y([n]; I) = \operatorname{Pf} \left( Y X {}^{t}Y \right),$$

where $I$ runs over all $n$-element subset of $[N] = \{1, 2, \dots, N\}$ and

$$X(I) = \left( x_{i_p, i_q} \right)_{1 \le p,\, q \le n}, \quad Y([n]; I) = \left( y_{p, i_q} \right)_{1 \le p,\, q \le n}$$

for $I = \{i_1 < i_2 < \cdots < i_n\}$.

To a partition $\lambda$ of length $\le n$, we associate

$$I_n(\lambda) = \{\lambda_1 + n, \lambda_2 + n - 1, \dots, \lambda_n + 1\}.$$

Then the correspondence $\lambda \mapsto I_n(\lambda)$ gives a bijection between partitions $\lambda \subset (m^n)$ and $n$-element subsets of $[m + n]$.

## Proof of the character identity (2/3)

Let $X$ be the skew-symmetric matrix with all $1$s above the diagonal, then

$$\operatorname{Pf} X(I) = 1$$

for any $n$-element subset $I$.

Let $Y$ be the $n \times (m+n)$ matrix with entries given by

$$y_{i,j} = \begin{cases} \dfrac{x_i^{j-n+k}}{\prod_{l=k+1}^{n}(1 - x_i^{-1} x_l)} - \dfrac{x_i^{-(j-n+k)}}{\prod_{l=k+1}^{n}(1 - x_i x_l)} & \text{if } 1 \le i \le k, \\ x_i^{j-1} & \text{if } k+1 \le i \le n. \end{cases}$$

Then the bialternant formula says

$$\operatorname{sp}_{\lambda}^{(k,n-k)}(\boldsymbol{x}) = \frac{\det Y([n]; I_n(\lambda))}{\det Y([n]; I_n(\emptyset))}.$$

## Proof of the character identity (3/3)

Hence we apply the minor summation formula to obtain

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_{\lambda}^{(k,n-k)}(\boldsymbol{x}) = \sum_{I} \operatorname{Pf} X(I) \cdot \frac{\det Y([n]; I)}{\det Y([n]; I_n(\emptyset))} = \frac{1}{\det Y([n]; I_n(\emptyset))} \operatorname{Pf}\left(Y X {}^{t}Y\right).$$

Now by evaluating $\operatorname{Pf}\left(Y X {}^{t}Y\right)$, we can show

$$\frac{1}{\det Y([n]; I_n(\emptyset))} \operatorname{Pf}\left(Y X {}^{t}Y\right) = \mathrm{o}^{B}_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2}.$$

## $q$-analogue of Hopkins' Conjecture (1/2)

For $\sigma \in \mathcal{A}^m(S(\delta_n + \delta_k))$, we define

$$w(\sigma) = kt_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k+1)t_l(\sigma),$$

$$v(\sigma) = \left(k - \frac{1}{2}\right) t_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k)t_l(\sigma),$$

where $t_l(\sigma) = \sum_i \sigma_{i,i+l}$ is the $l$th trace of $\sigma$.

**Proposition** For a fixed partition $\lambda$, the generating functions of shifted plane partitions of shape $\delta_n + \delta_k$ with profile $\lambda$ are given by

$$\sum_\sigma q^{w(\sigma)} = \mathrm{sp}_\lambda^{(k,n-k)}(q, q^2, \ldots, q^n),$$

$$\sum_\sigma q^{v(\sigma)} = \mathrm{sp}_\lambda^{(k,n-k)}(q^{1/2}, q^{3/2}, \ldots, q^{n-1/2}).$$

## $q$-analogue of Hopkins' Conjecture (2/2)

By specializing $x_i = q^i$ or $x_i = q^{i-1/2}$ in the character identity, we obtain

**Corollary**

$$\sum_{\sigma \in \mathcal{A}^m(S(\delta_n+\delta_k))} q^{w(\sigma)} = \frac{1}{q^{mk(k+1)/2}} \prod_{1 \le i \le j \le n} \frac{[m+i+j-1]}{[i+j-1]} \prod_{1 \le i \le j \le k} \frac{[m+i+j]}{[i+j]}.$$

$$\sum_{\sigma \in \mathcal{A}^m(S(\delta_n+\delta_k))} q^{v(\sigma)} = \frac{1}{q^{mk^2/2}} \prod_{i=1}^{n} \frac{[m/2+i-1/2]}{[i-1/2]} \prod_{1 \le i < j \le n} \frac{[m+i+j-1]}{[i+j-1]} \times \prod_{i=1}^{k} \frac{[m/2+i]}{[i]} \prod_{1 \le i < j \le k} \frac{[m+i+j]}{[i+j]}.$$

By putting $q = 1$, we have

**Corollary** (Hopkins Conjecture)

$$\# \mathcal{A}^m(S(\delta_n + \delta_k)) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

## Variations

**Theorem** Let $0 \le k \le n$. For a nonnegative even integer $m$, we have

$$\sum_{\lambda \subset (m^n):\text{even}} \mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$

$$= \mathrm{sp}_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2},$$

where the sum is taken over all partitions with $l(\lambda) \le n$ and $\lambda_1 \le m$ such that $\lambda_1, \dots, \lambda_m$ are even.

**Corollary** The number of shifted plane partitions $\sigma$ of shape $\delta_n + \delta_k$ such that $\sigma_{1,1}, \sigma_{2,2}, \dots$ are even is equal to

$$\prod_{1 \le i \le j \le n} \frac{m+i+j}{i+j} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

## Problem

The special cases $k = 0$ and $k = n$ of the character identity are

$$\sum_{\lambda \subset (m^n)} s_\lambda(x_1, \dots, x_n) = \mathrm{o}^B_{((m/2)^n)}(x_1, \dots, x_n) \cdot (x_1 \cdots x_n)^{m/2},$$

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_\lambda(x_1, \dots, x_n) = s_{(m^n)}(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1).$$

These identities describe the irreducible decompositions of the restriction from $\mathbf{Spin}_{2n+1}$ to $\mathbf{GL}_n$ and from $\mathbf{GL}_{2n+1}$ to $\mathbf{Sp}_{2n}$ respectively.

**Problem** Find a representation theoretical interpretation of the identity

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}^{(k,n-k)}_\lambda(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$

$$= \mathrm{o}^B_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2}.$$