Rouquier blocks

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Rouquier blocks / RoCK blocks

Symmetric groups

- Defined by Rouquier in 1998.
- In 2004, Chuang and Kessar proved Broué's abelian defect group conjecture for these blocks.
- In 2008, Chuang and Rouquier proved the conjecture for the symmetric group algebras.

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- Chuang and Kessar showed that if $w < p$ then the Rouquier block is Morita equivalent to the principal block of $\mathfrak{S}_p \wr \mathfrak{S}_w$.
- Chuang and Rouquier showed that every block of weight w is derived equivalent to a Rouquier block of weight w.

Rouquier blocks

Symmetric groups and Hecke algebras of type A

- Closed formula for the decomposition numbers when $w < p$.
- First examples of homomorphism spaces between Specht modules of dimension ≥ 2 when $p \neq 2$ discovered in Rouquier blocks.
- Classifying irreducible Specht modules when $p \neq 2$.
- Decomposition numbers for blocks of weight 3 are 0 or 1 if $p \geq 5$.
- James' Conjecture holds for Rouquier blocks, blocks of weight $w \leq 4$.

Rouquier blocks for other algebras

Examples

Rouquier blocks have also been defined for:

- Finite general linear groups (Miyachi, Turner)
- Other finite classical groups (Livesey)
- Chevalley groups of type E (Miyachi)
- Double covers of symmetric groups (Kleshchev and Livesey)

The Ariki-Koike algebras

Definition

Let $r \geq 1$ and $n \geq 0$. Let F be a field and $q \in F \setminus \{0\}$ and $\boldsymbol{Q} = (Q_1, \ldots, Q_r) \in F^r$.

The Ariki-Koike algebras

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$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0 && 1 \leq i \leq n-1, \\
T_i T_j &= T_j T_i && |i-j| > 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && 1 \leq i \leq n-2, \\
(T_0 - Q_1) \ldots (T_0 - Q_r) &= 0, && \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0. &&
\end{aligned}$$

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Note that if $r = 1$ then $T_0 = Q_1 \in F$ and $\mathcal{H}$ is the Hecke algebra of type A.

The Ariki-Koike algebras

Assumptions

Assume that q is a primitive eth root of unity for some $e \geq 2$. If $\boldsymbol{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{Z}^r$ define

$$\mathcal{H}_{r,n}(\boldsymbol{s}) = \mathcal{H}_{r,n}(q, (q^{s_1}, q^{s_2}, \ldots, q^{s_r})).$$

The Ariki-Koike algebras

Properties

$\mathcal{H}$ is a cellular algebra

$\rightsquigarrow$ Cell modules are indexed by r-multipartitions of n.

$\rightsquigarrow$ $S^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in \Lambda_n^r$ is called a Specht module.

$\rightsquigarrow$ All composition factors of $S^{\boldsymbol{\lambda}}$ belong to the same block.

Can partition the r-multipartitions into blocks.

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- $\rightsquigarrow$ All composition factors of $S^{\boldsymbol{\lambda}}$ belong to the same block.

Can partition the r-multipartitions into blocks.

Definition

Define an equivalence relation $\approx_e$ on $\Lambda^r \times \mathbb{Z}^r$ by saying that

$$(\boldsymbol{\lambda}, \boldsymbol{s}) \approx_e (\boldsymbol{\mu}, \boldsymbol{s}') \iff \boldsymbol{s} = \boldsymbol{s}',\ |\boldsymbol{\lambda}| = |\boldsymbol{\mu}| = n$$

and $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ lie in the same block of $\mathcal{H}_{r,n}(\boldsymbol{s})$.

The abacus

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots) \in \Lambda$ and $s \in \mathbb{Z}$. Define the β-set

$$B_s(\lambda) = \{\lambda_i - i + s \mid i \geq 1\}.$$

Combinatorics

The abacus

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots) \in \Lambda$ and $s \in \mathbb{Z}$. Define the β-set

$$B_s(\lambda) = \{\lambda_i - i + s \mid i \geq 1\}.$$

Let $e \geq 2$. Let $\mathscr{A}_e$ denote the set of abacus configurations on e runners. The e-abacus configuration of (λ, s) is given by putting a bead in position b for each $b \in B_s(\lambda)$.

Example of an abacus

Example

Recall that $B_s(\lambda) = \{\lambda_i + s - i \mid i \geq 1\}$. Let $s = 11$. Let

$$\lambda = (9, 9, 7, 3, 2, 2, 2, 1, 0, 0, 0, \ldots),$$
$$B_s(\lambda) = (19, 18, 15, 10, 8, 7, 6, 4, 2, 1, 0, \ldots).$$

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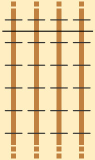

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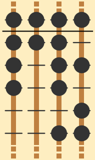
Let $e = 4$: 

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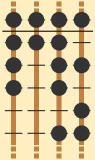
Let $e = 4$:  $\lambda_i = \#\{c < b_i \mid c \notin B\}$

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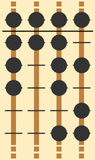
Let $e = 4$:  $\lambda_i = \#\{c < b_i \mid c \notin B\}$

Identifying with abacus configurations

We identify $\mathscr{A}_e$ with $\Lambda \times \mathbb{Z}$ and $\mathscr{A}_e^r$ with $\Lambda^r \times \mathbb{Z}^r$; then $\approx_e$ is an equivalence relation on $\mathscr{A}_e^r$. Call the equivalence classes blocks.

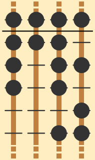
Cores and quotients

Definition

Define $\eta : \mathscr{A}_e \to \Lambda^e \times \mathbb{Z}^e$. If $(\lambda, s) \in \mathscr{A}_e$ then runner i gives a set of β-numbers corresponding to some $B_{t_i}(\sigma^i)$. Set

$$\eta(\lambda, s) = ((\sigma_0, \sigma_1, \ldots, \sigma_{e-1}), (t_0, t_1, \ldots, t_{e-1})).$$

Example



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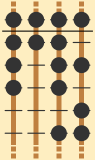

$$\eta(\lambda, s) = ((\emptyset, \emptyset, (1), (2^2, 1)), (3, 1, 4, 3))$$

Rouquier blocks

Definition

Say that $(\lambda, s) \in \mathscr{A}_e$ is a Rouquier partition if, given

$$\eta(\lambda, s) = ((\sigma_0, \sigma_1, \ldots, \sigma_{e-1}), (t_0, t_1, \ldots, t_{e-1}))$$

we have

$$t_{i+1} - t_i \geq |\boldsymbol{\sigma}| - 1 \text{ for all } 0 \leq i \leq e-2.$$

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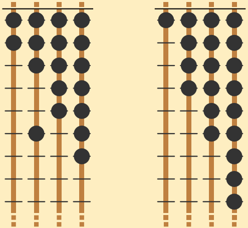
Definition

- $(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathscr{A}_e^r$ is a Rouquier multipartition if $(\lambda^{(k)}, s_k)$ is a Rouquier partition for all $1 \leq k \leq r$.
- A $\approx_e$-equivalance class $\mathscr{R} \subset \mathscr{A}_e^r$ is a Rouquier block if every $(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathscr{R}$ is a Rouquier multipartition.

Rouquier multipartitions

Example

Rouquier



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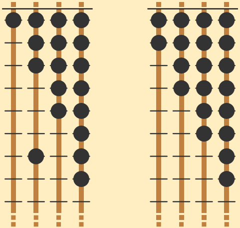
Not Rouquier



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Uglov's map

A pair of bijections

We define a pair of mutually inverse bijections

$$\Phi_r : \mathscr{A}_e \to \mathscr{A}_e^r, \qquad \Psi_r : \mathscr{A}_e^r \to \mathscr{A}_e.$$

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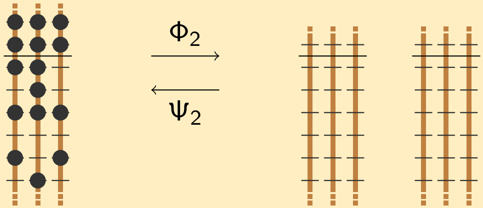

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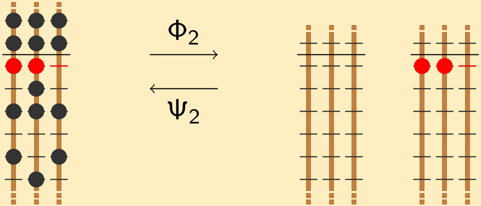

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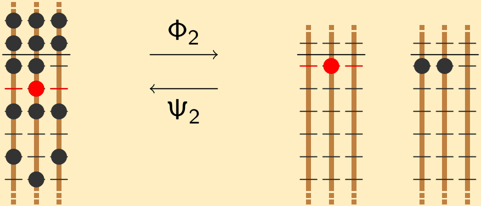

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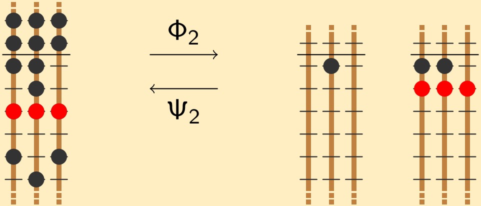

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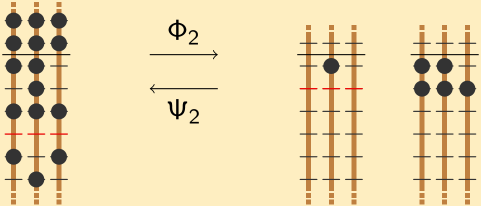

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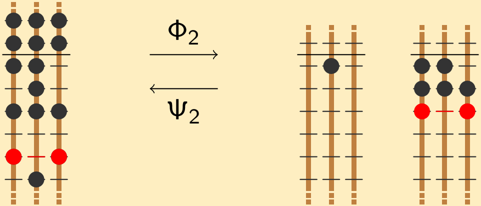

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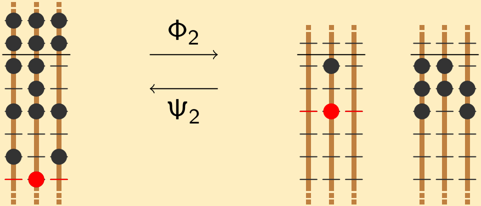

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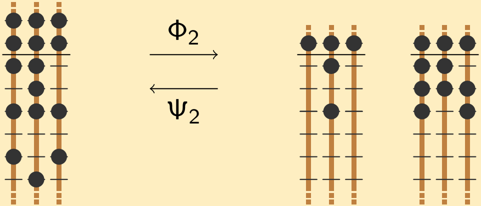

Ψ_r acting on blocks

Theorem

Suppose that $(\boldsymbol{\lambda}, \boldsymbol{s}), (\boldsymbol{\mu}, \boldsymbol{s}') \in \mathscr{A}_e^r$. Then

$$(\boldsymbol{\lambda}, \boldsymbol{s}) \approx_e (\boldsymbol{\mu}, \boldsymbol{s}') \implies \Psi_r(\boldsymbol{\lambda}, \boldsymbol{s}) \approx_e \Psi_r(\boldsymbol{\mu}, \boldsymbol{s}').$$

Φ_r acting on blocks

Recall

$$\mathscr{A}_e^r \xleftarrow{\quad\Phi_r\quad} \mathscr{A}_e \xrightarrow{\quad\eta\quad} \Lambda^e \times \mathbb{Z}^e \cong \mathscr{A}_r^e$$

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Theorem

Let $(\lambda, s), (\mu, s') \in \mathscr{A}_e$. Then

$$\Phi_r(\lambda, s) \approx_e \Phi_r(\mu, s') \iff \eta(\lambda, s) \approx_r \eta(\mu, s').$$

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Theorem

Let $(\lambda, s), (\mu, s') \in \mathscr{A}_e$. Then

$$\Phi_r(\lambda, s) \approx_e \Phi_r(\mu, s') \iff \eta(\lambda, s) \approx_r \eta(\mu, s').$$

Corollary

The map $\Psi_r \circ \eta$ gives a bijection between a block of $\mathscr{A}_e^r$ and a block of $\mathscr{A}_r^e$.

Generating Rouquier blocks

Definition

Say that $(\lambda, s) \in \mathscr{A}_e$ is a r-Rouquier partition if, given

$$\eta(\lambda, s) = ((\sigma_0, \sigma_2, \ldots, \sigma_{e-1}), (t_0, t_1, \ldots, t_{e-1}))$$

we have

$$t_{i+1} - t_i \geq |\boldsymbol{\sigma}| - r \text{ for all } 0 \leq i \leq e-2.$$

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Proposition

Suppose that $\mathscr{R} \subset \mathscr{A}_e$ is a r-Rouquier block. Then $\Phi_r(\mathscr{R}) \subset \mathscr{A}_e^r$ is a union of Rouquier blocks.

Decomposition Equivalence

Definition

Suppose that $\mathscr{R} \subset \mathscr{A}_e^r$ and $\mathscr{R}' \subset \mathscr{A}_{e'}^{r'}$ are two blocks. We say that $\mathscr{R}$ and $\mathscr{R}'$ are decomposition equivalent if there is a bijective map $\Upsilon : \mathscr{R} \to \mathscr{R}'$ such that for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{R}$,

- $\Upsilon(\boldsymbol{\lambda}) \trianglerighteq \Upsilon(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$.
- $\Upsilon(\boldsymbol{\lambda})$ indexes a simple module if and only if $\boldsymbol{\lambda}$ indexes a simple module.
- If $\boldsymbol{\lambda}$ indexes a simple module then

$$[S^{\boldsymbol{\mu}} : D^{\boldsymbol{\lambda}}]_{\mathscr{R}} = [S^{\Upsilon(\boldsymbol{\mu})} : D^{\Upsilon(\boldsymbol{\lambda})}]_{\mathscr{R}'}.$$

Scopes Equivalence

Theorem (Dell'Arciprete 2022)

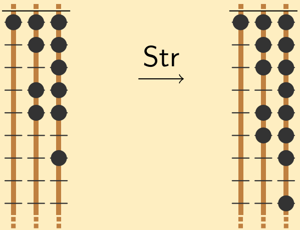
Suppose that $\mathscr{R}$ is a block of $\mathscr{A}_e^r$ such that no multipartition in $\mathscr{R}$ has addable i-nodes. Let Υ_i be the map that removes all possible i-nodes from a multipartition. Then $\mathscr{R}$ and $\Upsilon_i(\mathscr{R})$ are decomposition equivalent.

Stretching

Theorem

If $\mathscr{R} \subset \mathscr{A}_e^r$ is a Rouquier block which is not a core block then $\mathrm{Str}(\mathscr{R})$ is a Rouquier block and is decomposition equivalent to $\mathscr{R}$.

Example



Rouquier blocks

Theorem

Every Rouquier block $\mathscr{R} \subset \mathscr{A}_e^r$ is decomposition equivalent to a block $\mathscr{R}' \subset \mathscr{A}_e^r$ such that $\Psi_r(\mathscr{R}')$ is a subset of a Rouquier block.

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Open questions

- Can we say 'Morita equivalent' as well as 'decomposition equivalent' everywhere above?
- Assuming yes, are Rouquier blocks Morita equivalent to some 'local object'?
- Can we classify the Rouquier blocks up to decomposition / Morita equivalence?
- Is there a closed formula for the decomposition numbers of Rouquier blocks? - when $p = 0$ or the defect is small?