An example of A_2 Rogers-Ramanujan bipartition identities of level 3

(based on arXiv:2205.04811)

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OIST Representation Theory Seminar@online

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Plan

(1) Rogers-Ramanujan identities and the main results

(2) Representation theoretic background and related previous studies

(3) Comments on the proofs

In 1913, Ramanujan wrote a letter to Hardy.

Rogers-Ramanujan continued fraction

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{\ddots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

Hardy : ("The Indian Mathematician Ramanujan", Amer.Math.Month.**44**,1937),

"They defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

The RR continued fraction can be derived by the impressive formulas:

$$1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\cdots}$$

$$1+\sum_{n\geq 1}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\cdots}$$

Pochhammer symbols:

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$$(a_1,\cdots,a_m;q)_n := (a_1;q)_n\cdots(a_m:q)_n$$

Rogers-Ramanujan identities: $\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}, \quad \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}$

MacMahon: ("Combinatory Analysis", Cambridge University Press,1915),

"This most remarkable theorem has been verified as far as the coefficient of x^{89} by actual expansion so that there is practically no reason to doubt its truth; but it has not yet been established"

It is said that Ramanujan rediscovered Rogers's proof in the library (Hardy, "Ramanujan", CUP, 1940), but it may not be true (Sills,"An Invitation to the Rogers-Ramanujan Identities", CRS Press,2018).

Schur and MacMahon observed that the RR identities are equivalent to the following elementary statements:

① The partitions of n with condition (A) are equinumerous to those with (B).

② The partitions of n with condition (C) are equinumerous to those with (D).

(A): adjacent parts differ by $\geqq 2$ (B): each part is congruent to ± 1 mod 5

(C): (A) and each part is not equal to 1 (D): each part is congruent to ± 2 mod 5

Ex: ϕ 1 2 = 1+1 3 = 2+1 = 1+1+1

4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1

5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1

6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1
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= 1+1+1+1+1+1+1

Rogers-Ramanujan partition theorem: Consider the following conditions.

(R1) adjacent parts differ by ≧2

(R2) each part is not equal to 1

Let RR (resp. RR') be the set of partitions that satisfy (R1) (resp. (R1) and (R2)).

$$\Rightarrow \sum_{\lambda \in RR} q^{|\lambda|} = \frac{1}{(q, q^4; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n}, \quad \sum_{\lambda \in RR'} q^{|\lambda|} = \frac{1}{(q^2, q^3; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n}$$

Kanade-Russell conjecture (2014): Consider the following conditions.

(K1) differences are ≧3 at distance 2

(K2) if adjacent parts differ by ≦1, then their sum is divisible by 3

(K3) each part is not equal to 1

(K4) each part is not equal to 2

Let K (resp. K', K'') be the set of partitions that satisfies (K1)-(K2) (resp. (K1)-(K3), (K1)-(K4)). Then, ...

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$$\sum_{\lambda \in K} q^{|\lambda|} = \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty} = \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m (q^3;q^3)_n}$$

$$\sum_{\lambda \in K'} q^{|\lambda|} = \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty} = \sum_{m,n \geq 0} \frac{q^{m(m+1)+3mn+3n(n+1)}}{(q;q)_m (q^3;q^3)_n}$$

$$\sum_{\lambda \in K''} q^{|\lambda|} = \frac{1}{(q^4, q^3, q^6, q^5; q^9)_\infty} = \sum_{m,n \geq 0} \frac{q^{m(m+2)+3mn+3n(n+1)}}{(q;q)_m (q^3;q^3)_n}$$

* One can interpret the equalities partition theoretically. E.g.) 「The partitions of n in K are equinumerous to those whose parts are congruent to 1,3,6,8 mod 9」

The main result in this talk is described in terms of 2-color partitions.

Def: A 2-color partition of n is a partition of n whose parts are positive integers and colored positive integers.

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Rem A 2-color partition of n is identified with a **bipartition** of n.

Ex 4+4+3+2+1+1+1 ←→ ((4,2,1), (4,3,1,1))

Theorem [T]: Consider the following conditions on 2-color partitions.

(D1) adjacent parts differ by ≧2 if their sum is not divisible by 3 or their colors are the same

(D2) if adjacent parts differ by 2 and their sum is not divisible by 3, then their colors are not "colored and uncolored (in this order)"

(D3) does not contain (3k, 3k, 3k-2), (3k+2, 3k, 3k), (3k+2, 3k+1, 3k-1, 3k-2)

i.e., does not contain (3, 3, 1), (5, 3, 3), (5, 4, 2, 1) and their "3-step shifts"

(D4) each part is not equal to 1, 1, 2. Then, ...

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(D3) does not contain (3k, 3k, 3k-2), (3k+2, 3k, 3k), (3k+2, 3k+1, 3k-1, 3k-2)

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(D4) each part is not equal to 1, 1, 2.

Let R (resp. R') be the set of 2-color partitions with (D1)-(D3) (resp. (D1)-(D4)).

$$\Rightarrow \sum_{\lambda\in R} q^{|\lambda|} = \frac{(q^2,q^4;q^6)_\infty}{(q,q,q^3,q^3,q^5,q^5;q^6)_\infty} = \sum_{a,b,c,d\geq 0} \frac{q^{a(a+1)+b(b+2)+3c(c+1)+3d(d+1)+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

$$\sum_{\lambda\in R'} q^{|\lambda|} = \frac{1}{(q,q^2,q^2,q^3;q^6)_\infty} = \sum_{a,b,c,d\geq 0} \frac{q^{a^2+b^2+3c^2+3d^2+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

(Rem) As usual, one can get a "difference-mod partition interpretation" for R'.

Plan

(1) Rogers-Ramanujan identities and the main results

(2) Representation theoretic background and related previous studies

(3) Comments on the proofs

In the 1980s, Lepowsky-Wilson gave a Lie theoretic proof of the RR identities using $A^{(1)}_1$-modules $V(2\Lambda_0+\Lambda_1)$, $V(3\Lambda_0)$.

Ref: J.Lepowsky-R.Wilson, The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities, Invent.Math.,77, 199-290 (1984),...

The work is a continuation of the vertex operator construction of $V(\Lambda_0)$.

Ref: J.Lepowsky-R.Wilson, Construction of the affine Lie algebra $A^{(1)}_1$, Comm.Math.Phys.62 (1978), 43-53.

$$\mathrm{ch}V(2\Lambda_0+\Lambda_1)=\frac{1}{(q;q^2)_\infty}\frac{1}{(q,q^4;q^5)_\infty}=\frac{1}{(q;q^2)_\infty}\chi(V(2\Lambda_0+\Lambda_1))$$

$$\mathrm{ch}V(3\Lambda_0)=\frac{1}{(q;q^2)_\infty}\frac{1}{(q^2,q^3;q^5)_\infty}=\frac{1}{(q;q^2)_\infty}\chi(V(3\Lambda_0))$$

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The vertex operator construction was obtained independently by Frenkel-Kac and Segal in the context of string theory (dual resonance models). A motivation of Lepowsky-Wilson was the RR identities and later H.Garland pointed out a similarity can be seen in the dual resonance models. It is fair to say that the RR identities inspired representation theory of the affine Lie algebras.

Def: $\mathrm{Par} = \bigsqcup_{\ell \geq 0} \{\lambda = (\lambda_1, \cdots, \lambda_\ell) \in \mathbb{Z}^\ell \mid \lambda_1 \geq \cdots \lambda_\ell \geq 1\}$: the set of integer partitions

Notation: For $\lambda = (\lambda_1, \cdots, \lambda_\ell) \in \mathrm{Par}$

(1) λ_i : part, $\ell = \ell(\lambda)$: length, $|\lambda| := \lambda_1 + \ldots + \lambda_\ell$: size

(2) $m_j(\lambda) = \#\{ j \geqq 1 \mid \lambda_i = j \}$: multiplicity

Ex:

$$RR = \{\lambda \in \mathrm{Par} \mid \forall j \geq 1, m_j(\lambda) + m_{j+1}(\lambda) \leq 1\} \qquad RR' = RR \cap \{\lambda \in \mathrm{Par} \mid m_1(\lambda) = 0\}$$
$$= \{\lambda \in \mathrm{Par} \mid 1 \leq \forall i \leq \ell - 1, \lambda_i - \lambda_{i+1} \geq 2\}$$

Def: Two classes C, D $\subseteq$ Par are partition theoretically equivalent ($C \sim D$)

if $|C \cap \mathrm{Par}(n)| = |D \cap \mathrm{Par}(n)|$ for any $n \geqq 0$.

Here, $\mathrm{Par}(n) := \{ \lambda \in \mathrm{Par} \mid |\lambda| = n \}$ (the set of partitions of n).

Rogers-Ramanujan partition theorem: $RR \overset{\mathrm{PT}}{\sim} T^{(5)}_{1,4}, \quad RR' \overset{\mathrm{PT}}{\sim} T^{(5)}_{2,3}$

where $T^{(N)}_{a,b,\cdots} := \{\lambda \in \mathrm{Par} \mid 1 \leq \forall i \leq \ell(\lambda), \lambda_i \equiv a, b, \cdots \pmod{N}\}$

Andrews-Gordon identities

An analog for $V((2k-i)\Lambda_0+(i-1)\Lambda_1)$ is known.

Thm： Let $k \geqq 2$ and $1 \leqq i \leqq k$.

(1) $\{\lambda \in \mathrm{Par} \mid m_1(\lambda) < i \text{ and } \forall j, \lambda_j - \lambda_{j+k-1} \geq 2\} \overset{\mathrm{PT}}{\sim} T^{(2k+1)}_{\{1,\cdots,2k\}\setminus\{i,2k+1-i\}}$

(2) $\displaystyle\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}$ where $N_j := n_j+\cdots+n_{k-1}$

Rem： The case k=2, i=2,1 gives the RR partition theorem and the RR identities. It was proved by Gordon（1961）and Andrews（1974）combinatorially. A vertex operator proof was later given by Meurman-Primc (Adv.Math.1987). Even level analog is also known（Andrews-Gordon-Bressoud identities）.

Big Picture：For any affine Dynkin diagram and a dominant integral weight, there should exist an RR type identity whose infinite product is given by the principal character.

Toward $A^{(2)}_2$ Andrews-Gordon identities

In his Ph.D thesis (1992), S.Capparelli conjectured the following based on vertex operator calculations for level 3 $A^{(2)}_2$-modules $V(\Lambda_0+\Lambda_1)$ and $V(3\Lambda_0)$.

Thm : Let $C_a := RR \cap \{\lambda \mid m_a(\lambda) = 0 \text{ and } \forall j, \lambda_j - \lambda_{j+1} \leq 3 \Rightarrow \lambda_j + \lambda_{j+1} \in 3\mathbb{Z}\}$(a=1,2). Then,

$$C_a \overset{\text{PT}}{\sim} (\text{Strict} \cap \{\lambda \mid \forall j, \lambda_j \not\equiv \pm a \pmod 6\})$$

Rem : It was the first partition theorem coming from vertex operators.

Andrews proved by a q-series technique. Capparelli and Tamba-Xie proved by vertex operators. An Andrews-Gordon type identities are known [Takigiku-T].

Ex :

$$f_{C_a}(q) = \sum_{i,j,k\geq 0} \frac{q^{5i(i-1)/2+5j(j-1)/2+6k(k-1)+3ij+6ik+6jk+(3-a)i+(2+a)j+6k}}{(q^2;q^2)_i(q^2;q^2)_j(q^3;q^3)_k} = (-q^3,-q^6,-q^{3-a},-q^{3+a};q^6)_\infty$$

Big Picture : For any affine Dynkin diagram and a dominant integral weight, there should exist an RR type identity whose infinite product is given by the principal character.

Toward $A^{(2)}_2$ Andrews-Gordon identities

In his Ph.D thesis (2014), D.Nandi conjectured 3 partition theorems based on vertex operator calculations for level 4 $A^{(2)}_2$-modules $V(2\Lambda_1)$, $V(2\Lambda_0+\Lambda_1)$, $V(4\Lambda_0)$.

$$\mathcal{N}_1 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,3,4,10,11,12}, \quad \mathcal{N}_2 \overset{\mathrm{PT}}{\sim} T^{(14)}_{1,4,6,8,10,13}, \quad \mathcal{N}_3 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,5,6,8,9,12}$$

It was proved in [Takigiku-T, arXiv:1910.12461].

The Andrews-Gordon type identities are also known.

Ex :

$$f_{\mathcal{N}_1}(q) = \sum_{i,j\geq 0} (-1)^j \frac{q^{i(i+1)/2+j^2+2ij}}{(q;q)_i(q^2;q^2)_j} = \frac{1}{(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty}$$

Some results are known in the case of $A^{(2)}_2$ level ≧5. For example, on the result

$$\{\lambda = (a_1 \leq a_2 \leq \cdots) \mid a_1 \geq 2, a_2 \geq a_1, a_3 - a_2 \geq 2, a_4 \geq a_3\} \overset{\mathrm{PT}}{\sim} T^{(16)}_{2,3,4,5,11,12,13,14}$$

by M. Hirschhorn（1979）, the infinite product coincides with the principal character of a level 5 module, but the difference conditions “varies at places”.

Similar results are known for level 7 due to Caparelli (2004).

Theorem [Takigiku-T,arXiv:1910.12461, conjectured by Nandi]

Let $\mathcal{N}$ *denote the set of partitions* λ *satisfying the conditions* (N1)-(N6):

(N1) *For all* $1 \leq i \leq \ell(\lambda) - 1$, $\lambda_i - \lambda_{i+1} \neq 1$,

(N2) *For all* $1 \leq i \leq \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} \geq 3$,

(N3) *For all* $1 \leq i \leq \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,

(N4) *For all* $1 \leq i \leq \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} = 3$ *and* $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$,

(N5) *For all* $1 \leq i \leq \ell(\lambda) - 2$,
$\lambda_i - \lambda_{i+2} = 4$ *and* $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$ *and* $\lambda_{i+1} \neq \lambda_{i+2}$,

(N6) $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{\ell(\lambda)-1} - \lambda_{\ell(\lambda)})$ *does not match* $(3, 2^*, 3, 0)$.
Here 2^* *denotes any number (possibly zero) of repetitions of* 2.

Define $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \subseteq \mathcal{N}$ *by*

$$\mathcal{N}_1 = \{\lambda \in \mathcal{N} \mid m_1(\lambda) = 0\},$$

$$\mathcal{N}_2 = \{\lambda \in \mathcal{N} \mid m_i(\lambda) \leq 1 \text{ for } i = 1, 2, 3\},$$

$$\mathcal{N}_3 = \left\{ \lambda \in \mathcal{N} \,\middle|\, \begin{array}{c} m_1(\lambda) = m_3(\lambda) = 0, \; m_2(\lambda) \leq 1, \\ \forall k \geq 1, \; \lambda \text{ does not match } (2k+3, 2k, 2k-2, \cdots, 4, 2) \end{array} \right\}.$$

Then

$$\mathcal{N}_1 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,3,4,10,11,12}, \quad \mathcal{N}_2 \overset{\mathrm{PT}}{\sim} T^{(14)}_{1,4,6,8,10,13}, \quad \mathcal{N}_3 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,5,6,8,9,12}.$$

Toward $A^{(2)}_2$ Andrews-Gordon identities

In his Ph.D thesis (2014), D.Nandi conjectured 3 partition theorems based on vertex operator calculations for level 4 $A^{(2)}_2$-modules $V(2\Lambda_1)$, $V(2\Lambda_0+\Lambda_1)$, $V(4\Lambda_0)$.

$$\mathcal{N}_1 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,3,4,10,11,12}, \quad \mathcal{N}_2 \overset{\mathrm{PT}}{\sim} T^{(14)}_{1,4,6,8,10,13}, \quad \mathcal{N}_3 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,5,6,8,9,12}$$

It was proved in [Takigiku-T, arXiv:1910.12461].

The Andrews-Gordon type identities are also known.

Ex:

$$f_{\mathcal{N}_1}(q) = \sum_{i,j\geq 0} (-1)^j \frac{q^{i(i+1)/2+j^2+2ij}}{(q;q)_i(q^2;q^2)_j} = \frac{1}{(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty}$$

Some results are known in the case of $A^{(2)}_2$ level ≧5. For example, on the result

$$\{\lambda = (a_1 \leq a_2 \leq \cdots) \mid a_1 \geq 2, a_2 \geq a_1, a_3 - a_2 \geq 2, a_4 \geq a_3\} \overset{\mathrm{PT}}{\sim} T^{(16)}_{2,3,4,5,11,12,13,14}$$

by M. Hirschhorn (1979), the infinite product coincides with the principal character of a level 5 module, but the difference conditions “varies at places”.

Similar results are known for level 7 due to Caparelli (2004).

Toward $A^{(2)}_2$ Andrews-Gordon identities

In [Takigiku-T, Proc.AMS (2021)], RR type identities whose infinite products coincide with principal characters of level 5 and 7 modules were obtained.

Ex :
$$\sum_{i,j,k\geq 0} (-1)^k \frac{q^{i(i+1)/2+j^2+k^2+2ij+2ik+4jk+j}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{(q^2,q^3,q^4,q^5,q^{11},q^{12},q^{13},q^{14};q^{16})_\infty}$$

On level 6, McLaughlin-Sills（2008）gave "RR type" identities. For example,

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} = \frac{1}{(q^2,q^3,q^4,q^5,q^6,q^{12},q^{13},q^{14},q^{15},q^{16};q^{18})_\infty}$$

Recently,Kanade-Russell [Adv.Math.(2022)] gives "RR type" identities for any level that includes the identities of McLaughlin-Sills.

The case $A^{(2)}_{odd}$ and level 2

The case $A^{(2)}_5$ and level 2: Göllnitz-Gordon partition theorem (1960s) can be regarded a prediction of "Big Picture".

RR type partition theorem: $\{\lambda \in \mathrm{Par} \mid \forall i, \lambda_i - \lambda_{i+1} \geq 2 \ \ (\geq \text{ is } > \text{ if } \lambda_i \in 2\mathbb{Z})\} \overset{\mathrm{PT}}{\sim} T^{(8)}_{1,4,7}$

RR type identities: $\sum_{i,j\geq 0} \frac{q^{i(3i-1)/2+4j^2+4ij}}{(q;q)_i(q^4;q^4)_j} = \frac{1}{(q,q^4,q^7;q^8)_\infty}$ [Kurşungöz, JCTA, (2019)]

A relation to vertex operators was given by Kanade（Ramanujan J. (2018)）.

The case $A^{(2)}_7$ and level 2: RRPT can be regarded a prediction of "Big Picture".

A relation to vertex operators were studied by Misra-Bos（Commun.Alg. (1994)）

The $A^{(2)}_9$ and level 2: Kanade-Russell（Electron. J. Combin. (2019)）gave conjectures. Subsequently, Bringmann et.al.（Crelle, (2020)）and Rosengren（Ramanujan J.(2021)）proved. For example,

The case $A^{(2)}_{odd}$ and level 2

RR type PT: $\{\lambda \in \mathrm{Par} \mid {}^{m_1(\lambda)=0 \text{ and } \forall j\geq 0, m_{2j+1}(\lambda)\leq 1 \text{ and } \forall i, \lambda_i - \lambda_{i+1}\neq 1}_{\forall i, \lambda_i - \lambda_{i+2}\geq 4 \text{ if } \lambda_{i+1}\in 2\mathbb{Z} \text{ and } (\lambda_{i+1}=\lambda_{i+2} \text{ or } \lambda_i = \lambda_{i+1})}\} \overset{\mathrm{PT}}{\sim} T^{(12)}_{1,4,6,8,11}$

RR type identities: $\displaystyle\sum_{i,j,k,\geq 0} (-1)^k \frac{q^{(i+2j+3k)(i+2j+3k-1)/2+3k^2+i+6j+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k} = \frac{1}{(q,q^4,q^6,q^8,q^{11};q^{12})_\infty}$

The case $A^{(2)}_{11}$ and level 2: The principal characters coincide with those of $A^{(2)}_{2}$ level 4. Relations to vertex operators are not known.

The case $A^{(2)}_{13}$ and level 2: In [Takigiku-T, Proc.AMS (2021)], Andrews-Gordon type identities were conjectured.

conjecture: $\displaystyle\sum_{i,j,k,\geq 0} (-1)^j \frac{q^{i(i+1)/2+j(j+2)+4k(2k+1)+2ij+4ik+4jk}}{(q;q)_i (q^2;q^2)_j (q^4;q^4)_k} = \frac{1}{(q,q^4,q^6,q^8,q^{10},q^{12},q^{15};q^{16})_\infty}$

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It is expected that Kanade-Russell conjecture is the case of $D^{(3)}_4$ and level 3.

Plan

(1) Rogers-Ramanujan identities and the main results

(2) Representation theoretic background and related previous studies

(3) Comments on the proofs

A standard q-series technique to prove Andrews-Gordon identities

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty} \text{ where } N_j := n_j+\cdots+n_{k-1}$$

is the Bailey Lemma.

Bailey Lemma (simple ver) : If $\alpha = (\alpha_n)_{n\geq 0}, \beta = (\beta_n)_{n\geq 0}$ form a Bailey pair, i.e.,

$$\forall n \geq 0, \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}$$

Then, $\alpha' = (\alpha'_n := a^n q^{n^2} \alpha_n)_{n\geq 0}, \beta' = (\beta'_n := \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q;q)_{n-j}} \beta_j)_{n\geq 0}$ also form a Bailey pair.

One can find a nice survey in ¥S3.1 in “*An Invitation to the Rogers-Ramanujan identities*” by A.Sills (CRC Press). In the paper [Andrews-Schilling-Warnaar, *An A_2 Bailey lemma and Rogers-Ramanujan-type identities*, J.Amer.Math.Soc.(1999)] the authors showed “Bailey Lemma” which gives RR type identities for $A^{(1)}_2$.

Andrews-Schilling-Warnaar (the case of level 3) :

$$\sum_{s,t\geq 0} \frac{q^{s^2-st+t^2}(q^3;q^3)_{s+t}}{(q;q)^2_{s+t}(q^3;q^3)_s(q^3;q^3)_t} = \frac{1}{(q;q)_\infty}\frac{(q^2,q^4;q^6)_\infty}{(q,q,q^3,q^3,q^5,q^5;q^6)_\infty} = \frac{1}{(q;q)_\infty}\chi(V(\Lambda_0+\Lambda_1+\Lambda_2))$$

$$\sum_{s,t\geq 0} \frac{q^{s^2-st+t^2+s+t}(q^3;q^3)_{s+t}}{(q;q)_{s+t+1}(q;q)_{s+t}(q^3;q^3)_s(q^3;q^3)_t} = \frac{1}{(q;q)_\infty}\frac{1}{(q^2,q^3,q^3,q^4;q^6)_\infty} = \frac{1}{(q;q)_\infty}\chi(V(2\Lambda_0+\Lambda_1))$$

Due to the factor $(q;q)_\infty$, the infinite sums above are not manifestly positive.

Especially, it seems difficult to interpret them representation theoretically.

Thm [T] :

$$\sum_{\lambda\in R} q^{|\lambda|} = \frac{(q^2,q^4;q^6)_\infty}{(q,q,q^3,q^3,q^5,q^5;q^6)_\infty} = \sum_{a,b,c,d\geq 0} \frac{q^{a(a+1)+b(b+2)+3c(c+1)+3d(d+1)+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

$$\sum_{\lambda\in R'} q^{|\lambda|} = \frac{1}{(q,q^2,q^2,q^3;q^6)_\infty} = \sum_{a,b,c,d\geq 0} \frac{q^{a^2+b^2+3c^2+3d^2+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

In the paper, Andrews-Schilling-Warnaar gave RR type identities whose infinite sums are manifestly positive and whose infinite products are some of level 4 $A^{(1)}_2$ standard modules. Such identities seems to be called A_2 RR identities.

Ex: A_2 RR identities (Andrews-Schilling-Warnaar (1999), level 4)

$$\sum_{r,s\geq 0}\frac{q^{r^2-rs+s^2}(q;q)_{2r}}{(q;q)_r(q;q)_{2r-s}(q;q)_s}=\frac{1}{(q,q,q^3,q^4,q^6,q^6;q^7)_\infty}=\chi(2\Lambda_0+\Lambda_1+\Lambda_2)$$

Another A_2 RR identities were obtained by Corteel-Welsh (Ann.Comb.(2019)) for level 4 and Corteel-Dousse-Uncu (Proc.Amer.Math.Soc.(2022)) for level 5. Recently, Warnaar (arXiv:2111.07550) and Kanade-Russell (arXiv:2203.05690) discuss on the general level. In the case of level divisible by 3, it seems that little is known and the result below is the only known A_2 RR identities of level 3.

Thm [T]:
$$\sum_{\lambda\in R}q^{|\lambda|}=\frac{(q^2,q^4;q^6)_\infty}{(q,q,q^3,q^3,q^5,q^5;q^6)_\infty}=\sum_{a,b,c,d\geq 0}\frac{q^{a(a+1)+b(b+2)+3c(c+1)+3d(d+1)+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

$$\sum_{\lambda\in R'}q^{|\lambda|}=\frac{1}{(q,q^2,q^2,q^3;q^6)_\infty}=\sum_{a,b,c,d\geq 0}\frac{q^{a^2+b^2+3c^2+3d^2+2ab+3ac+3ad+3bc+3bd+6cd}}{(q;q)_a(q;q)_b(q^3;q^3)_c(q^3;q^3)_d}$$

In the paper, Andrews-Schilling-Warnaar gave RR type identities whose infinite sums are manifestly positive and whose infinite products are some of level 4 $A^{(1)}_2$ standard modules. Such identities seems to be called A_2 RR identities.

A brief survey of [T, arXiv:2205.04811]

(1) For an affine GCM $X^{(r)}_N$ and a dominant integral weight λ, calculate a conjectural spanning (monomial) vectors by the vertex operators. They are parameterized by certain m-color partitions, where

m = (the number of roots of X_N) /(r-twisted Coxeter number).

Although (in the principal picture) this calculation was done only for $A^{(1)}_1$ (any level), $A^{(2)}_2$ (level 2,3,4) and $A^{(1)}_2$ level 3 (the case in this talk), it seems that the same method can be applicable in any case in principle.

---

(2) Calculate q-diff. equations for 2 variable generating functions of R and R'.

Ex : $f_R(x,q) = \sum_{\lambda \in R} x^{\ell(\lambda)} q^{|\lambda|}$

$$
\begin{aligned}
&(2+3xq^4+xq^6)f_{\mathsf{R}}(x,q) \\
&= (2+4xq+4xq^2+4xq^3+3xq^4+xq^6+4x^2q^3+6x^2q^4+6x^2q^5+8x^2q^6+2x^2q^7+2x^2q^8+2x^2q^9+6x^3q^7+2x^3q^9+3x^3q^{10}+x^3q^{12})f_{\mathsf{R}}(xq^3,q) \\
&- x^2q^7(2+2q+3xq+4xq^2+xq^3+4xq^4-xq^5+4xq^6+xq^7+2x^2q^5+6x^2q^7+6x^2q^8+2x^2q^9+2x^2q^{10}+4x^2q^{11}+3x^3q^9+x^3q^{11} \\
&+6x^3q^{12}+2x^3q^{14})f_{\mathsf{R}}(xq^6,q)+x^4q^{21}(1-xq^6)^2(2+3xq+xq^3)f_{\mathsf{R}}(xq^9,q),
\end{aligned}
$$

This can be done automatically by a generalization of Andrews' linked partition ideals using finite automata [Takigiku-T, arXiv:1910.12461] for a wide class of partitions.

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\begin{aligned}
&(2 + 3xq^4 + xq^6) f_{\mathsf{R}}(x,q) \\
&= (2 + 4xq + 4xq^2 + 4xq^3 + 3xq^4 + xq^6 + 4x^2q^3 + 6x^2q^4 + 6x^2q^5 + 8x^2q^6 + 2x^2q^7 + 2x^2q^8 + 2x^2q^9 + 6x^3q^7 + 2x^3q^9 + 3x^3q^{10} + x^3q^{12}) f_{\mathsf{R}}(xq^3, q) \\
&- x^2q^7(2 + 2q + 3xq + 4xq^2 + xq^3 + 4xq^4 - xq^5 + 4xq^6 + xq^7 + 2x^2q^5 + 6x^2q^7 + 6x^2q^8 + 2x^2q^9 + 2x^2q^{10} + 4x^2q^{11} + 3x^3q^9 + x^3q^{11} \\
&+ 6x^3q^{12} + 2x^3q^{14}) f_{\mathsf{R}}(xq^6, q) + x^4q^{21}(1 - xq^6)^2(2 + 3xq + xq^3) f_{\mathsf{R}}(xq^9, q),
\end{aligned}
$$

This can be done automatically by a generalization of Andrews' linked partition ideals using finite automata [Takigiku-T, arXiv:1910.12461] for a wide class of partitions.

(3) For Andrews-Gordon type series we want to prove, enhance x suitably.

$$\sum_{a,b,c,d\geq 0} \frac{q^{a^2+b^2+3c^2+3d^2+2ab+3ac+3ad+3bc+3bd+6cd} x^{a+b+2c+2d}}{(q;q)_a (q;q)_b (q^3;q^3)_c (q^3;q^3)_d}$$

$$\sum_{a,b,c,d\geq 0} \frac{q^{a(a+1)+b(b+2)+3c(c+1)+3d(d+1)+2ab+3ac+3ad+3bc+3bd+6cd} x^{a+b+2c+2d}}{(q;q)_a (q;q)_b (q^3;q^3)_c (q^3;q^3)_d}$$

Then, calculate q-diff. equations and verify that they are equal to $f_R(x,q)$, $f_{R'}(x,q)$

This can be done automatically by q-version of Sister Celine's technique, which is used to obtain a closed form for a hypergeometric summation. E.g., $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(4) Change the x-enhancement from a+b+2c+2d to a+b+c+2d and check that the q-diff equations are equal to those of 2 variable generating function of cylindric partitions of profile (1,1,1), (3,0,0) after a multiplication of $(xq;q)_\infty$.

(3) For Andrews-Gordon type series we want to prove, enhance x suitably.

$$\sum_{a,b,c,d\geq 0} \frac{q^{a^2+b^2+3c^2+3d^2+2ab+3ac+3ad+3bc+3bd+6cd} x^{a+b+2c+2d}}{(q;q)_a (q;q)_b (q^3;q^3)_c (q^3;q^3)_d}$$

$$\sum_{a,b,c,d\geq 0} \frac{q^{a(a+1)+b(b+2)+3c(c+1)+3d(d+1)+2ab+3ac+3ad+3bc+3bd+6cd} x^{a+b+2c+2d}}{(q;q)_a (q;q)_b (q^3;q^3)_c (q^3;q^3)_d}$$

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(4) Change the x-enhancement from a+b+2c+2d to a+b+c+2d and check that the q-diff equations are equal to those of 2 variable generating function of cylindric partitions of profile (1,1,1), (3,0,0) after a multiplication of $(xq;q)_\infty$.

(5) For a (usual 1-variable) generating function of cylindric partitions, an infinite product expansion is known [Bordin, Duke.Math.(2007)].

(Summary) Vertex operators (representation theory) (*automatic part)

→ defining conditions for 2-color partitions R and R'

→ q-diff. eq.s for 2-variable generationg functions $f_R(x,q)$ and $f_{R'}(x,q)$

= q-diff. eq.s for Andrews-Gordon type series with x-enhancement

→ q-diff. eq.s with different x-enhancement

= q-diff. eq.s for 2-var. gen. func.s of cylindric partitions ▪ $(xq;q)_\infty$(Corteel-Welsh)

→ forget the enhancement（i.e., x=1）and apply Bordin product formula

Expectation : For any level of $A^{(1)}_2$, it is expected that RR type identities should exist as “Big Picture”. In that case, 2-color partitions play a role rather than partitions.

Thank you!

(Summary)　Vertex operators (representation theory)　　(*automatic part)

→　defining conditions for 2-color partitions R and R’

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→　forget the enhancement（i.e., x=1）and apply Bordin product formula

【Q and A】

On representation theoretic interpretation

For RR identities (due to Lepowsky-Wilson, Meurman-Primc) : For

$$\mathfrak{g}(A_1^{(1)}) = \widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

we have a basis { B(n), X(n'), c, d | n : odd , n'∈Z } (principal realization), where

$$B(n) = (e+f) \otimes t^n, \quad X(2k) = h \otimes t^{2k}, \quad X(2k+1) = (f-e) \otimes t^{2k+1}.$$

Then, the following is a basis of V((i+1)Λ_0 + (2-i)Λ_1), for i = 1, 2.

$$\{B(-\mu_1) \cdots B(-\mu_{\ell'}) X(-\lambda_1) \cdots X(-\lambda_\ell) v\}.$$

Here, v is a corresponding highest weight vector, $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ runs in RR (resp. RR') for i=1 (resp. i=2) and $\mu = (\mu_1, \ldots, \mu_\ell)$ runs in odd partitions.

Recall:

$$\sum_{\lambda \in RR} q^{|\lambda|} = \frac{1}{(q, q^4; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n}, \qquad \sum_{\lambda \in RR'} q^{|\lambda|} = \frac{1}{(q^2, q^3; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n}$$

$$\mathrm{ch} V(2\Lambda_0 + \Lambda_1) = \frac{1}{(q;q^2)_\infty} \frac{1}{(q, q^4; q^5)_\infty} = \frac{1}{(q;q^2)_\infty} \chi(V(2\Lambda_0 + \Lambda_1))$$

$$\mathrm{ch} V(3\Lambda_0) = \frac{1}{(q;q^2)_\infty} \frac{1}{(q^2, q^3; q^5)_\infty} = \frac{1}{(q;q^2)_\infty} \chi(V(3\Lambda_0))$$

On representation theoretic interpretation

In the case of $A^{(1)}_2$ level 3 ([T, arXiv:2205.04811]): For

$$\mathfrak{g}(A_2^{(1)}) = \widehat{\mathfrak{sl}_3} = \mathfrak{sl}_3 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

we have a basis { B(n), X_+(n'), X_-(n'), c, d | n : not divisible by 3, n' ∈ Z } , where we omit explicit description for B(n) and $X_\pm$(n') (principal realization). Then,

$$\{B(-\mu_1)\cdots B(-\mu_{\ell'})X_{\mathrm{color}(\lambda_1)}(-\mathrm{cont}(\lambda_1))\cdots X_{\mathrm{color}(\lambda_\ell)}(-\mathrm{cont}(\lambda_\ell))v\}$$

is a basis of V($(2i-1)\Lambda_0 + (2-i)\Lambda_1 + (2-i)\Lambda_2$) for i = 1, 2. Here, v is a corresponding highest weight vector, λ= ($\lambda_1,\dots,\lambda_\ell$) runs in R (resp. R') for i = 1 (resp. i=2) and μ= ($\mu_1,\dots,\mu_\ell$) runs in 3-class regular partitions (i.e., partitions no parts are divisible by 3). For a (usual) integer n, we define color(n) = +, cont(n) = n and for a colored integer n, we define color(n) = -, cont(n) = n.

Recall:

$$\mathrm{ch}V(\Lambda_0+\Lambda_1+\Lambda_2) = \frac{1}{(q,q^2;q^3)_\infty}\frac{(q^2,q^4;q^6)_\infty}{(q,q,q^3,q^3,q^5,q^5;q^6)_\infty}, \quad \mathrm{ch}V(3\Lambda_0) = \frac{1}{(q,q^2;q^3)_\infty}\frac{1}{(q^2,q^3,q^3,q^4;q^6)_\infty}$$