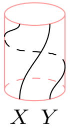
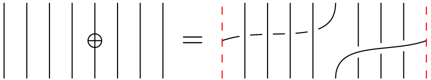
Affinization of monoidal categories

(Monoidal categories on cylinders)





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Preprint: arXiv:2010.13598 (joint with Youssef Mousaaid)

Outline

Goal: Guided by intuition of monoidal categories on cylinders, formalize the notion of affinization of a monoidal category.

Overview:

1. String diagrams for strict monoidal categories
2. Examples
3. Definition of affinization
4. Dot presentation
5. Examples
6. Horizontal and vertical trace

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

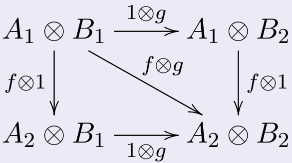
Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:



Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbb{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

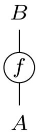
Example (Associative algebras)

A (strict) $\Bbbk$-linear monoidal category with one object is simply a commutative associative unital $\Bbbk$-algebra.

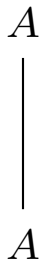
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



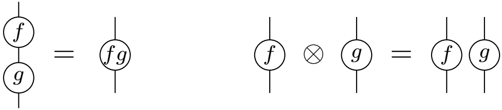
The identity map $1_A\colon A \to A$ is a string with no label:



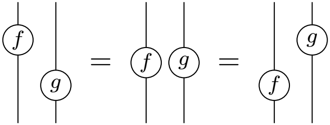
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

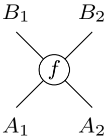
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Example: monoidally generated symmetric groups

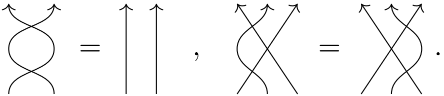
Define a strict monoidal category $\mathcal{S}ym$ with one generating object $\uparrow$ and denote

$$1_\uparrow = \Big\uparrow$$

We have one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}ym}(\uparrow^{\otimes n}) = S_n$$

is the symmetric group on n letters.

Example: monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

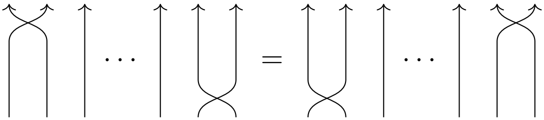
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j| > 1$$

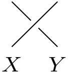
for simple transpositions follows for free from the interchange law:

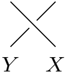


Note: If we define $\mathcal{Sym}$ to be $\Bbbk$-linear, then $\mathrm{End}_{\mathcal{Sym}}(\uparrow^{\otimes n}) = \Bbbk S_n$.

Braidings

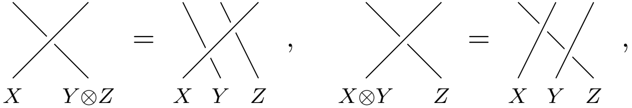
A strict monoidal category $\mathcal{C}$ is braided if it is equipped with isomorphisms

$\beta_{X,Y} =$ 

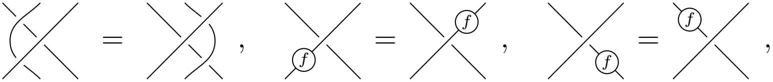
$: X \otimes Y \to Y \otimes X, \qquad \beta^{-1}_{X,Y} =$ 

,

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, satisfying



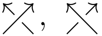
and



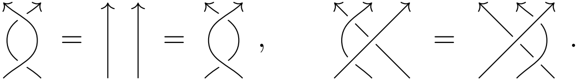
for all morphisms f in $\mathcal{C}$ (and any compatible labelling of strands).

Braid groups

The category *Braid* of braids over the disc is isomorphic to the strict monoidal category generated by a single object $\uparrow$, and morphisms

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow,$

subject to the relations



Universal property: *Braid* is the free braided monoidal category generated by a single object.

Note that

$$\mathrm{End}_{\mathcal{B}raid}(\uparrow^{\otimes n}) = \text{braid group of type } A_{n-1}.$$

Let $\mathcal{B}raid_{\Bbbk}$ be the $\Bbbk$-linearization of $\mathcal{B}raid$, i.e. the strict $\Bbbk$-linear monoidal category with the same generators and relations.

Fix $z \in \Bbbk^\times$, and let $\mathcal{H}$ be obtained from $\mathcal{B}raid_{\Bbbk}$ by imposing the Conway skein relation

$$\overset{\nwarrow\nearrow}{\times} - \overset{\nwarrow\nearrow}{\times} = z \uparrow\uparrow .$$

Then

$$\mathrm{End}_{\mathcal{H}}(\uparrow^{\otimes n}) = \text{Iwahori–Hecke algebra of type } A_{n-1}.$$

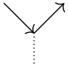
(Often one sets $z = q - q^{-1}$ for some $q \in \Bbbk^\times$.)

Duals

Suppose a strict monoidal category $\mathcal{C}$ has two objects $\uparrow$ and $\downarrow$, with

$$1_\uparrow = \Big\uparrow \quad , \quad 1_\downarrow = \Big\downarrow .$$

A morphism $\mathbb{1} \to \downarrow \otimes \uparrow$ would have string diagram

 , where $= 1_{\mathbb{1}}$.

We typically omit the dotted line and draw:

 $: \mathbb{1} \to \downarrow \otimes \uparrow .$

Similarly, we can have

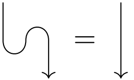
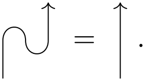
 $: \uparrow \otimes \downarrow \to \mathbb{1}.$

Duals

We say that $\downarrow$ is right dual to $\uparrow$ (and $\uparrow$ is left dual to $\downarrow$) if there exist morphisms

 $: \mathbb{1} \to \downarrow \otimes \uparrow.$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}.$

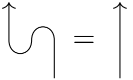
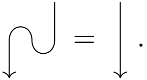
such that

 and 

Objects $\uparrow$ and $\downarrow$ are both left and right dual to each other if we also have

 $: \mathbb{1} \to \uparrow \otimes \downarrow$ and  $: \downarrow \otimes \uparrow \to \mathbb{1}$

such that

 and 

We say $\mathcal{C}$ is rigid if all objects have left and right duals.

Duals: example

Let $\Bbbk$ be a field and consider the category $\mathrm{Vect}_{\Bbbk}$ of f.d. $\Bbbk$-vector spaces.

Unit object: $\Bbbk$

Fix a f.d. $\Bbbk$-vector space V.

Claim: The dual vector space V^* is both left and right dual to V, in the sense mentioned above.

Proof: Fix a basis B of V. Let $\{\delta_v : v \in B\}$ be the dual basis of V^*. Viewing V and $\uparrow$ and V^* as $\downarrow$, we define

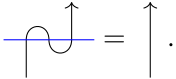
 $: \Bbbk \to V^* \otimes V, \qquad 1 \mapsto \sum_{v\in B} \delta_v \otimes v,$

 $: V \otimes V^* \to \Bbbk, \qquad v \otimes f \mapsto f(v),$

 $: \Bbbk \to V \otimes V^*, \qquad 1 \mapsto \sum_{v\in B} v \otimes \delta_v,$

 $: V^* \otimes V \to \Bbbk, \qquad f \otimes v \mapsto f(v).$

Let's check the relation



The left-hand side is the composition

$$V \cong V \otimes \Bbbk \xrightarrow{1_V \otimes \cup} V \otimes V^* \otimes V \xrightarrow{\cap \otimes 1_V} \Bbbk \otimes V \cong V,$$

$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(w) \otimes v \mapsto \sum_{v \in V} \delta_v(w) v = w.$$

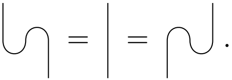
The verification of the other relations is analogous.

Self-dual objects

An object I is self-dual if there exist morphisms

$$\bigcup : \mathbb{1} \to \mathsf{I} \otimes \mathsf{I}. \qquad \text{and} \qquad \bigcap : \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}.$$

such that



Example

In the category of finite-dimensional inner product spaces (over $\mathbb{R}$ or $\mathbb{C}$), all objects V are self-dual.

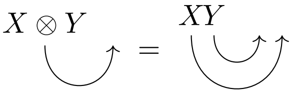
$$\bigcap : V \otimes V \to \Bbbk, \qquad v \otimes w \mapsto \langle v, w \rangle,$$
$$\bigcup : \Bbbk \to V \otimes V, \qquad 1 \mapsto \textstyle\sum_{v \in B} v \otimes v,$$

where B is an orthonormal basis for V.

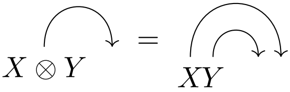
A strict monoidal category $\mathcal{C}$ is a strict pivotal category if every object X has a right dual $X^\vee$ and the following conditions are satisfied:

1. For all objects X and Y in $\mathcal{C}$,

$$(X^\vee)^\vee = X, \quad (X \otimes Y)^\vee = Y^\vee \otimes X^\vee, \quad \mathbb{1}^\vee = \mathbb{1},$$

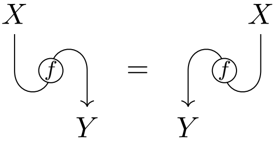


and



.

2. For every morphism $f \colon X \to Y$ in $\mathcal{C}$,

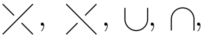


$: Y^\vee \to X^\vee.$

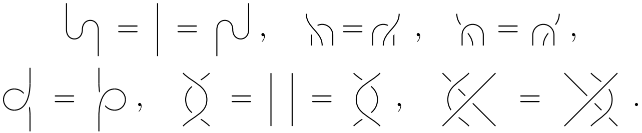
Intuition: In a strict pivotal category, morphisms are invariant under isotopy.

Framed tangles

The category $\mathcal{FT}$ of framed tangles over the disc is generated by a single object I and morphisms



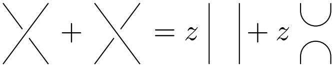
subject to the relations



Universal property: $\mathcal{FT}$ is the free ribbon category generated by a self-dual object.

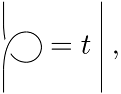
Kauffman skein category

Fix $z, t \in \Bbbk^\times$. The Kauffman skein category $\mathcal{KS}(z,t)$ is the category obtained from $\mathcal{FT}_\Bbbk$ by imposing the relations



(Kauffman skein relation)

and



$$\bigcirc = \left(1 - \frac{t + t^{-1}}{z}\right) 1_{\mathbb{1}}.$$

Then

$$\mathrm{End}_{\mathcal{KS}(z,t)}(\mathsf{I}^{\otimes n}) \cong \text{Kauffman tangle algebra}$$
$$\cong \text{Birman–Murakami–Wenzl (BMW) algeba.}$$

Viewing a link as an element of $\mathrm{End}_{\mathcal{KS}(z,t)}(\mathbb{1})$ gives rise to its Kauffman polynomial.

Affinization

Definition

The affinization of a strict monoidal category $\mathcal{C}$ is the category $\mathrm{Aff}(\mathcal{C})$ obtained from $\mathcal{C}$ by adjoining invertible morphisms

$$\xi_{X,Y} \colon X \otimes Y \to Y \otimes X, \quad X, Y \in \mathrm{Ob}(\mathcal{C}),$$

subject to the relations

$$\xi_{X,Y\otimes Z} = \xi_{Z\otimes X,Y} \circ \xi_{X\otimes Y,Z}, \quad \xi_{X_2,Y_2} \circ (g \otimes f) = (f \otimes g) \circ \xi_{X_1,Y_1}.$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2)$, and $g \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$.

Picture morphisms in $\mathrm{Aff}(\mathcal{C})$ as $\mathcal{C}$-diagrams on the cylinder, with

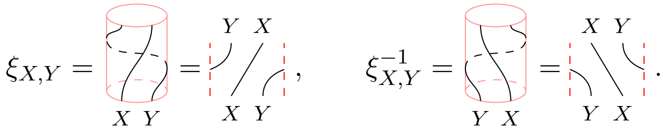
$\xi_{X,Y} =$ 

, $\xi^{-1}_{X,Y} =$ 

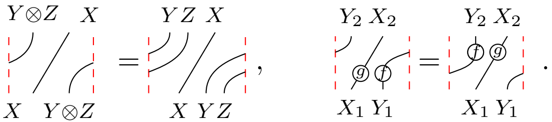
. "coils"

Affinization

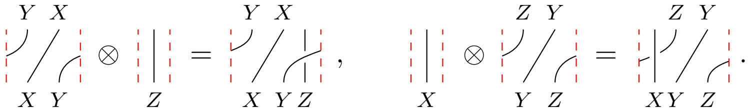
Cutting open the cylinder, we draw



Then the defining relations become

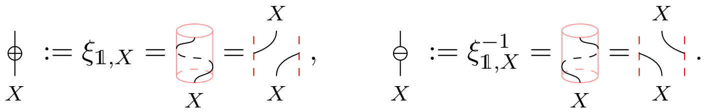


If $\mathcal{C}$ is braided, then $\mathrm{Aff}(\mathcal{C})$ is strict monoidal, with tensor product given by nesting cylinders. For example,

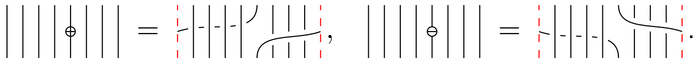


Dot presentation

Suppose $\mathcal{C}$ is braided, so that $\mathrm{Aff}(\mathcal{C})$ is strict monoidal. Define

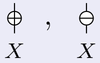


Thus

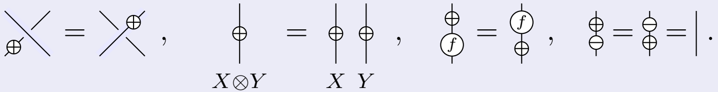


Theorem

The affinization $\mathrm{Aff}(\mathcal{C})$ is obtained from $\mathcal{C}$ by adjoining morphisms



$: X \to X, \quad X \in \mathrm{Ob}(\mathcal{C}),$ subject to relations



Pivotal structures

Suppose $\mathcal{C}$ is braided and rigid/pivotal. Then $\mathrm{Aff}(\mathcal{C})$ is also rigid/pivotal.

Self-dual case (unoriented)

If X is self-dual, for strands labelled X we have

Non-self-dual case (oriented)

Define the invertible dots

Then we have

Examples: Towers of algebras

Suppose $\mathcal{C}$ is a strict linear monoidal category such that

- objects are generated by a single object X,
- $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m}) = 0$ when $m \neq n$.

The collection $\mathcal{C}(n) := \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$, $n \in \mathbb{N}$, is a tower of algebras.

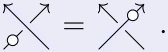
Braids

Category $\mathcal{B}raid$ of braids over the disc has

generators  relations 

Affine braids

The affinization $\mathrm{Aff}(\mathcal{B}raid)$ is the category of braids over the annulus. Dot presentation obtained from $\mathcal{B}raid$ by adding

invertible generator , relation 

Examples: Towers of algebras

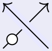
Hecke algebras

The category $\mathcal{H}$ corresponding to the tower of Hecke algebras of type A is the $\Bbbk$-linearization of $\mathcal{B}raid$ modulo the Conway skein relation

 $-$  $= (q - q^{-1}) \uparrow\uparrow, \quad q \in \Bbbk^\times.$

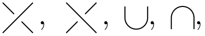
Affine Hecke algebras

The affinization $\mathrm{Aff}(\mathcal{H})$ corresponds to the tower of affine Hecke algebras of type A. Obtained from $\mathcal{H}$ by adding

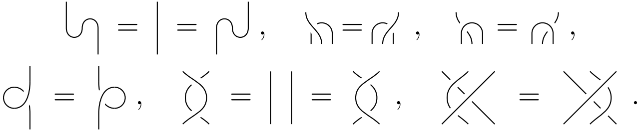
invertible generator , relation  $=$  .

Examples: Tangles

The category $\mathcal{FT}$ of framed tangles over the disc is generated by a single object I and morphisms

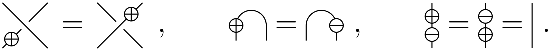


subject to the relations



The affinization $\mathrm{Aff}(\mathcal{FT})$ is the category of framed tangles over the annulus.

$\mathrm{Aff}(\mathcal{FT})$ is obtained from $\mathcal{FT}$ adding morphisms $\oplus$, $\ominus$ and relations



Similar results hold for non-framed tangles and oriented (framed and non-framed) tangles.

Examples: Skein categories

Kauffman skein category

The affinization of the Kauffman skein category over the disc is the Kauffman skein category over the annulus.

Dot presentation matches the affine Kauffman skein category of Gao–Rui–Song.

Temperley–Lieb category

The affinization of the Temperley–Lieb category is the affine Temperley–Lieb category.

Actions

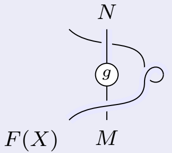
Suppose $\mathcal{C}, \mathcal{M}$ are braided strict pivotal categories and $F \colon \mathcal{C} \to \mathcal{M}$ is a monoidal functor.

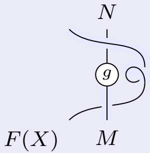
Then $\mathcal{C}$ acts on $\mathcal{M}$ via the action

$$X \cdot M = F(X) \otimes M, \quad f \cdot g = F(f) \otimes g.$$

Theorem

The above can be extended to an action of $\mathrm{Aff}(\mathcal{C})$ on $\mathcal{M}$ with

$\xi_X \cdot g =$ 

, $\xi_X^{-1} \cdot g =$ 

.

Note: More generally, have action whenever $\mathcal{C}$ is a strict monoidal category and $\mathcal{M}$ is a balanced strict monoidal category.

Horizontal trace

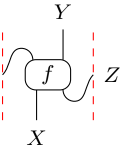
Suppose $\mathcal{C}$ is a strict monoidal category. Fix $X, Y \in \mathrm{Ob}(\mathcal{C})$ and consider pairs

$$(Z, f), \quad Z \in \mathrm{Ob}(\mathcal{C}), \quad f \colon X \otimes Z \to Z \otimes Y.$$

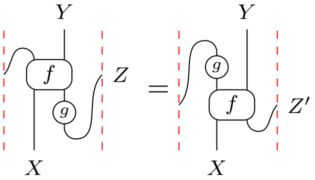
Define an equivalence relation on such pairs by

$$(Z, f \circ (1_X \otimes g)) \sim (Z', (g \otimes 1_Y) \circ f),$$
$$g \in \mathrm{Hom}_{\mathcal{C}}(Z, Z'), \quad f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Z', Z \otimes Y).$$

Picture the equivalence class $[Z, f]$ as the cylindrical diagram



, so that

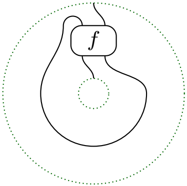


.

Important: The cups and caps above have no precise meaning!

Horizontal trace

Also sometimes see $[Z, f]$ as the annular diagram



.

Important (again): The cups and caps above have no precise meaning!

Can define a natural composition, turning the horizontal trace $\mathrm{Tr}_{\mathrm{h}}(\mathcal{C})$ into a category.

If $\mathcal{C}$ is braided, then the horizontal trace is a monoidal category.

Horizontal trace

Theorem

If $\mathcal{C}$ is a rigid strict monoidal category then $\mathrm{Aff}(\mathcal{C})$ is isomorphic to the horizontal trace $\mathrm{Tr_h}(\mathcal{C})$ of $\mathcal{C}$.

In general (i.e. when $\mathcal{C}$ is not rigid) the affinization and the horizontal trace are different.

It is the affinization, not the horizontal trace, that naturally corresponds to $\mathcal{C}$-diagrams on the cylinder.

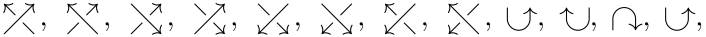
Example

Recall that $\mathrm{Aff}(\mathcal{B}\mathit{raid})$ is the category of annular braids.

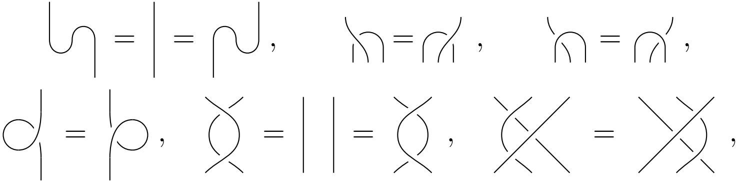
However, the strings in $\mathrm{Tr_h}(\mathcal{B}\mathit{raid})$ can only wrap in one direction around the cylinder, and one also has closed components wrapping around the cylinder.

Final example: HOMFLYPT skein category

The category $\mathcal{FOT}$ of framed oriented tangles over the disc is isomorphic to the strict monoidal category generated by objects $\uparrow$, $\downarrow$, and morphisms



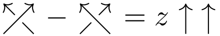
subject to the relations



for all orientations of the strands.

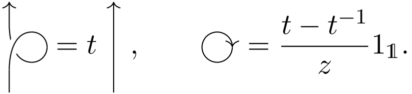
Final example: HOMFLYPT skein category

Work over $\Bbbk = \mathbb{Z}[z, z^{-1}, t, t^{-1}]$. The HOMFLYPT skein category $\mathcal{OS}(z,t)$ over the disc is the category obtained from $\mathcal{FOT}_{\Bbbk}$ by imposing the relations



(Conway skein relation)

and



Given an oriented link diagram L, we have

$$t^{-\text{writhe}(L)} L = H(L) \frac{t - t^{-1}}{z} 1_{\mathbb{1}},$$

where

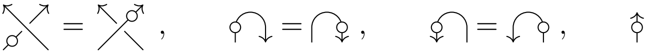
- writhe(L) is the writhe number (# pos crossings $-$ # neg crossings),
- $H(L)$ is the HOMFLYPT polynomial of L.

The affinization $\mathrm{Aff}(\mathcal{OS}(z,t))$ is isomorphic to the HOMFLPYT skein category over the annulus.

Dot presentation matches the affine oriented skein category of Brundan.

Affinization $\mathrm{Aff}(\mathcal{OS}(z,t))$ is obtained from $\mathcal{OS}(z,t)$ by adjoining morphisms

subject to the relations



is invertible.