Defining an Affine Partition Algebra

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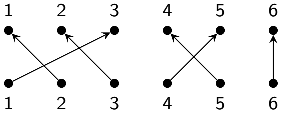
Symmetric Group

- $\mathbb{C}S_n$ group algebra of the symmetric group over $\mathbb{C}$
- Let $s_i = (i, i+1)$ be the simple transpositions exchanging i and $i+1$
- S_n has a presentation via generators s_i for $1 \leq i < n$, and relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad |i-j| > 1$$

- View permutations diagrammatically:

$(1,3,2)(4,5)(6) =$



$\in S_6$

Jucys-Murphy (JM) Elements (1/2)

- The JM elements $X_1, \ldots, X_n$ of $\mathbb{C}S_n$ are given by

$$X_j = \sum_{i=1}^{j-1} (i, j).$$

- These satisfy (for $j \neq i-1, i$)

$$X_i X_j = X_j X_i, \qquad X_j s_i = s_i X_j, \qquad X_{j+1} = s_j X_j s_j + s_j.$$

- Recursively we may define

$$X_{j+1} = s_j X_j s_j + s_j$$

by setting $X_1 = 0$.

Jucys-Murphy (JM) Elements (2/2)

- $\langle X_1, \ldots, X_n \rangle \subset \mathbb{C}S_n$ is a maximal commutative subalgebra.
- Let $\{S^\lambda \mid \lambda \in \Lambda_n\}$ be a complete set of simple $\mathbb{C}S_n$.
- By the Wedderburn-Artin Theorem with $n_\lambda = \dim(S^\lambda)$,

$$\mathbb{C}S_n \cong \prod_{\lambda \in \Lambda_n} M_{n_\lambda}(\mathbb{C}).$$

- Picking a certain basis for each S^λ, we may realise

$$\langle X_1, \ldots, X_n \rangle \cong \prod_{\lambda \in \Lambda_n} Dia_{n_\lambda}(\mathbb{C}) \subset \prod_{\lambda \in \Lambda_n} M_{n_\lambda}(\mathbb{C}).$$

- Understanding the diagonal entires under this isomorphism (i.e. eigenvales of the JM elements) allows one to recover most of the combinatorial information of the representation theory of $\mathbb{C}S_n$.
- Lastly $Z(\mathbb{C}S_n) = \mathrm{Sym}[X_1, \ldots, X_n]$.

Schur-Weyl Duality (1/2)

- Let $V = \mathrm{Span}_{\mathbb{C}}\{v_1, \dots, v_n\}$ and consider the Lie algebra $\mathfrak{gl}_n(V)$
- $V^{\otimes k}$ is an $\mathfrak{gl}_n(V)$-module via the diagonal action
- For $a = (a_1, \dots, a_k) \in [n]^k$ let

$$v_a := v_{a_1} \otimes \cdots \otimes v_{a_k} \in V^{\otimes k}$$

- For any $\pi \in S_k$ let $\pi(a) := (a_{\pi^{-1}(1)}, \dots, a_{\pi^{-1}(k)})$
- We have an algebra isomorphism

$$\psi_{n,k} : \mathbb{C}S_k \to \mathrm{End}_{\mathfrak{gl}_n(V)}(V^{\otimes k})$$

given by $\psi_{n,k}(\pi)(v_a) = v_{\pi(a)}$.

Schur-Weyl Duality (2/2)

- Let $E_{a,b}$ be the matrix with (a, b) entry 1, and 0 elsewhere
- Let $E_{a,b}^{(i)} \in \mathrm{End}(V^{\otimes k})$ act on the i-th factor of $V^{\otimes k}$ by $E_{a,b}$.
- Define

$$\Omega_{i,j} = \sum_{a,b=1}^{n} E_{a,b}^{(i)} \otimes E_{b,a}^{(j)} \in \mathrm{End}(V^{\otimes k})$$

- Then we have that $\psi_{n,k}(s_i) = \Omega_{i,i+1}$
- Also we have that

$$\psi_{n,k}(X_j) = \sum_{i=1}^{j-1} \Omega_{i,j}$$

Degenerate Affine Hecke Algebra $\mathcal{H}_k$

- Going from $\mathbb{C}S_k$ to $\mathcal{H}_k$ we replace X_j with a formal variable x_i.
- $\mathcal{H}_k = \langle s_i, x_j \mid i \in [k-1], j \in [k] \rangle$ with defining relations

  (i) $s_i^2 = 1$
  (ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
  (iii) $s_i s_j = s_j s_i, \quad |i-j| > 1$
  (iv) $x_i x_j = x_j x_i$
  (v) $x_{i+1} = s_i x_i s_i + s_i$
  (vi) $s_i x_j = x_j s_i$, for $j \neq i-1, i$

- $\mathbb{C}S_k$ may be realised as a quotient of $\mathcal{H}_k$ via $x_j \mapsto X_j$
- $\mathcal{H}_k$ has a basis given by $\{x_1^{a_1} \cdots x_k^{a_k} \pi \mid a_i \in \mathbb{Z}_{\geq 0}, \pi \in S_k\}$.
- Hence have the subalgebras $\mathbb{C}S_k$, $\mathbb{C}[x_1, \ldots, x_k] \subset \mathcal{H}_k$.
- $Z(\mathcal{H}_k) = \mathrm{Sym}[x_1, \ldots, x_k]$.

Extended Schur-Weyl Duality

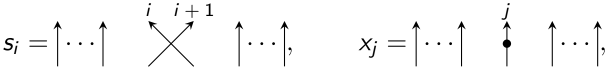
- Let M be any $\mathfrak{gl}_n(V)$-module.
- We consider $M \otimes V^{\otimes k}$ as a $\mathfrak{gl}_n(V)$-module via the diagonal action.
- We say that M is in the 0-th factor $M \otimes V^{\otimes k}$.
- We have a representation $\psi_{n,k}^{(M)} : \mathcal{H}_k \to \mathrm{End}_{\mathfrak{gl}_n(V)}(M \otimes V^{\otimes k})$ by

$$\psi_{n,k}^{(M)}(s_i) = \Omega_{i,i+1}, \quad \text{and} \quad \psi_{n,k}^{(M)}(x_j) = \sum_{i=0}^{j-1} \Omega_{i,j}$$

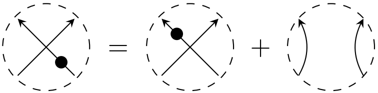
- $\psi_{n,k}^{(M)}(x_j)$ differs to $\psi_{n,k}(X_j)$ by the term $\Omega_{0,j}$

Diagrammatics

- We can also extend the diagrammatics to $\mathcal{H}_k$ as follows:



and we impose the local relation



which captures the relations $s_i x_{i+1} = x_i s_i + 1$.

- These diagrammatics may be realised within the Heisenberg Category which we will introduce later.

Affinization

- This "affinization" process of introducing new commuting variables as generators which mimic the JM-elements has been successfully applied to other diagram algabras:
  - Brauer Algebra
  - Walled-Brauer Algebra
  - Braid Group Algebra
- In some of these cases new central genators are introduced alongside the new variables to account for closed loops
- We seek to employ this process to the partition algebra $\mathcal{A}_{2k}$ to construct an affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$.
- We expect $\mathcal{A}_{2k}^{\text{aff}}$ to uphold analogous results to those of $\mathcal{H}_k$.

Partition Algebra (1/2)

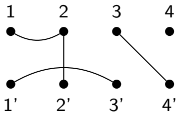
- Let Π_{2k} be the set of all set partitions of $\{1, \ldots, k\} \cup \{1', \ldots, k'\}$
- For z a formal variable, the partition algebra has basis

$$\mathcal{A}_{2k} = \text{Span}_{\mathbb{C}[z]}\{\alpha \mid \alpha \in \Pi_{2k}\}$$

$$\mathcal{A}_{2k-1} = \text{Span}_{\mathbb{C}[z]}\{\alpha \mid \alpha \in \Pi_{2k},\ k \sim k'\}$$

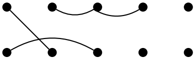
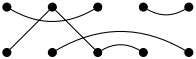
- We view such basis elements diagrammatically: For $k = 4$,

$$\{\{1, 2, 2'\}, \{1', 3'\}, \{3, 4'\}, \{4\}\} =$$

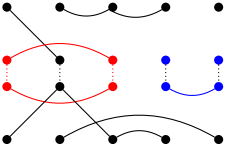
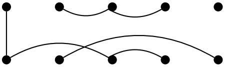


Partition Algebra (2/2)

- We explain multiplication through example: For $k = 5$,

$\alpha =$  $\beta =$ 

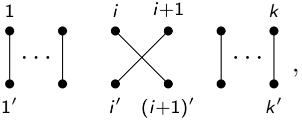
Then the product is given by

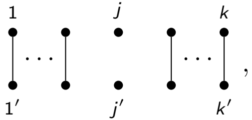
$\alpha\beta =$  $= z^{1+1}$ 

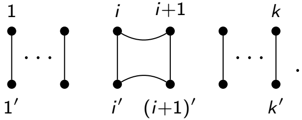
- We have that $\mathbb{C}S_k$ is a subalgebra of $\mathcal{A}_{2k}$
- We let $\mathcal{A}_{2k}(\delta) := \mathcal{A}_{2k}/(z - \delta)$ for any $\delta \in \mathbb{C}$.

Standard Generators

- The following elements generate $\mathcal{A}_{2k}$:

$s_i =$ 

$e_{2j-1} =$ 

$e_{2i} =$ 

- A presentation consisting of 16 relations of $\mathcal{A}_{2k}$, in terms of these generators, was first presented by Halverson and Ram.

Jucys-Murphy elements of $\mathcal{A}_{2k}$ (1/3)

- The JM elements $L_1, \ldots, L_{2k}$ of the partition algebra $\mathcal{A}_{2k}$ are given recursively by setting $L_1 = 0$, $L_2 = e_1$, then

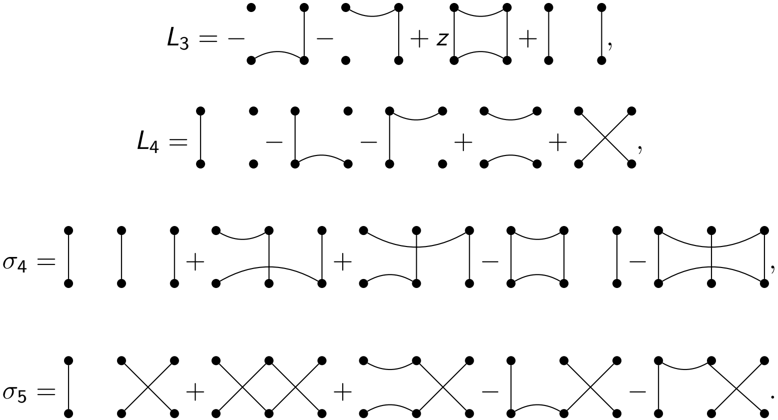
$$L_{2i+1} = s_i L_{2i-1} s_i - L_{2i} e_{2i} - e_{2i} L_{2i} + (z - L_{2i-1}) e_{2i} + \sigma_{2i}$$
$$L_{2i+2} = s_i L_{2i} s_i - s_i L_{2i} e_{2i} - e_{2i} L_{2i} s_i + e_{2i} L_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1}$$

- Where $\sigma_2 = 1$, $\sigma_3 = s_1$, and recursively

$$\begin{aligned}\sigma_{2i} &= s_{i-1} s_i \sigma_{2i-2} s_i s_{i-1} + e_{2i-2} L_{2i-2} s_i e_{2i-2} s_i + s_i e_{2i-2} L_{2i-2} s_i e_{2i-2} \\ &\quad - e_{2i-2} L_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} - s_i e_{2i-2} e_{2i-1} e_{2i} s_{i-1} L_{2i-2} e_{2i-2} s_i. \\ \sigma_{2i+1} &= s_{i-1} s_i \sigma_{2i-1} s_i s_{i-1} + s_i e_{2i-2} L_{2i-2} s_i e_{2i-2} s_i + e_{2i-2} L_{2i-2} s_i e_{2i-2} \\ &\quad - s_i e_{2i-2} L_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} - e_{2i-2} e_{2i-1} e_{2i} s_{i-1} L_{2i-2} e_{2i-2} s_i.\end{aligned}$$

Jucys-Murphy elements of $\mathcal{A}_{2k}$ (2/3)

The first few non-trivial examples are



Jucys-Murphy elements of $\mathcal{A}_{2k}$ (3/3)

- $\mathcal{A}_{2k} = \langle \sigma_i, e_j \mid i,j \in [2k-1] \rangle$
- Enyang gave an alternative presentation of $\mathcal{A}_{2k}$ in these generators which has 20 defining relations
- It proved easier for us to work with the normalisations

$$t_{2i} := \sigma_{2i} - e_{2i}, \qquad t_{2i+1} := \sigma_{2i+1} - e_{2i},$$

$$X_i := \begin{cases} z - 1 - L_i, & \text{if } i \text{ odd} \\ L_i - 1, & \text{if } i \text{ even} \end{cases}$$

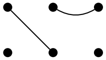
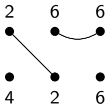
- $\langle X_1, \dots, X_{2k} \rangle \subset \mathcal{A}_{2k}$ is a maximal commutative subalgebra
- $Z(\mathcal{A}_{2k}) = \mathrm{SSym}[X_1, \dots, X_{2k}]$ which is generated by

$$X_1^n + X_3^n + \cdots + X_{2k-1}^n - (X_2^n + X_4^n + \cdots + X_{2k}^n)$$

for each $n \geq 0$.

Schur-Weyl Duality (1/3)

- We say a pair of tuples $(a, b) \in [n]^k \times [n]^k$ colours $\alpha \in \Pi_{2k}$ if:
    - Colouring vertex i with a_i, and i' with b_i, gives a coloured graph where vertices of the same block are the same colour
- We write $(a, b) \hookrightarrow \alpha$ when (a, b) colours α
- For example, letting $n = 6$ and $k = 3$,

$((2,6,6),(4,2,6)) \hookrightarrow$  $\longleftrightarrow$ 

Schur-Weyl Duality (2/3)

- Let $V = \text{Span}_{\mathbb{C}}\{v_1, \ldots, v_n\}$ be the permutation module of $\mathbb{C}S_n$
- $V^{\otimes k}$ is a $\mathbb{C}S_n$-module via the diagonal action
- There exists a surjective homomorphism

$$\psi_{n,2k} : \mathcal{A}_{2k} \to \text{End}_{S_n}(V^{\otimes k})$$

given by $z \mapsto n$ and

$$\psi_{n,2k}(\alpha)(v_b) = \sum_{\substack{a \in [n]^k \\ (a,b) \hookrightarrow \alpha}} v_a$$

for any $\alpha \in \Pi_{2k}$ and $v_b \in V^{\otimes k}$

- The map $\psi_{n,2k}$ realises an isomorphism $\mathcal{A}_{2k}(n) \cong \text{End}_{S_n}(V^{\otimes k})$ if and only if $n \geq 2k$

Schur-Weyl Duality (3/3)

- The action of the JM elements is given by

$$\psi_{n,2k}(X_{2r-1})(v_a) = \sum_{\substack{b=1 \\ b\neq a_r}}^{n} (a_r, b)(v_{a_1} \otimes \cdots \otimes v_{a_{r-1}}) \otimes v_{a_r} \otimes \cdots \otimes v_{a_k}$$

$$\psi_{n,2k}(X_{2r})(v_a) = \sum_{\substack{b=1 \\ b\neq a_r}}^{n} (a_r, b)(v_{a_1} \otimes \cdots \otimes v_{a_r}) \otimes v_{a_{r+1}} \otimes \cdots \otimes v_{a_k}$$

- The action of Enyang's generators is given by

$$\psi_{n,2k}(t_{2r})(v_a) = (1-\delta_{a_r,a_{r+1}})(a_r, a_{r+1})(v_{a_1} \otimes \cdots \otimes v_{a_{r-1}}) \otimes v_{a_r} \otimes \cdots \otimes v_{a_k}$$

$$\psi_{n,2k}(t_{2r+1})(v_a) = (1-\delta_{a_r,a_{r+1}})(a_r, a_{r+1})(v_{a_1} \otimes \cdots \otimes v_{a_{r+1}}) \otimes v_{a_{r+2}} \otimes \cdots \otimes v_{a_k}$$

Affine Partition Algebra (1/2)

- We construct $\mathcal{A}_{2k}^{\text{aff}}$ by replacing X_i with new commuting variables x_i, and introducing new central generators z_l to account for decorated floating components.
- We used Enyang's presentation, however we replace the generators t_i with new generators τ_i.
- So we have

$$\mathcal{A}_{2k}^{\text{aff}} = \langle e_i, \tau_i, x_j, z_l \mid i \in [2k-1], j \in [2k], l \in \mathbb{N} \rangle$$

  and there are 31 defining relations.
- 20 of these defining relations are τ counterparts to Enyang's presentation, while the remain relations are the ones picked to retain.

Affine Partition Algebra (2/2)

- We have a surjective homomorphism $\mathcal{A}_{2k}^{\text{aff}} \to \mathcal{A}_{2k}$ via

$$x_i \mapsto X_i, \quad \tau_i \mapsto t_i, \quad z_l \mapsto z(z-1)^l$$

- Both $\mathcal{A}_{2k}$ and $\mathbb{C}[x_1, \ldots, x_{2k}]$ are subalgebras of $\mathcal{A}_{2k}^{\text{aff}}$
- $\text{SSym}[x_1, \ldots, x_{2k}] \subset Z(\mathcal{A}_{2k}^{\text{aff}})$
- We have a surjective homomorphism $\mathcal{A}_{2k}^{\text{aff}} \to \mathcal{H}_k \otimes \mathcal{H}_k$ via

$$x_{2i-1} \mapsto -1 \otimes x_i, \quad x_{2i} \mapsto x_i \otimes 1, \quad e_i \mapsto 0$$

$$\tau_{2i} \mapsto 1 \otimes s_i, \quad \tau_{2i+1} \mapsto s_i \otimes 1, \quad z_l \mapsto 0$$

Extended Schur-Weyl Duality

- Let $M = \mathrm{Span}_{\mathbb{C}}\{m_1, \dots, m_d\}$ be any $\mathbb{C}S_n$-module
- We view $M \otimes V^{\otimes k}$ as an $\mathbb{C}S_n$-module via the diagonal action
- We have a algebra homomorphism

$$\psi^{(M)}_{n,2k} : \mathcal{A}^{\mathrm{aff}}_{2k} \to \mathrm{End}_{S_n}(M \otimes V^{\otimes k})$$

which extends $\psi_{n,2k}$ by having

$$\psi_{n,2k}(x_{2r})(m_i \otimes v_a) = \sum_{\substack{b=1 \\ b \neq a_r}}^{n} (a_r, b)(m_i \otimes v_{a_1} \otimes \cdots \otimes v_{a_r}) \otimes v_{a_{r+1}} \otimes \cdots \otimes v_{a_k}$$

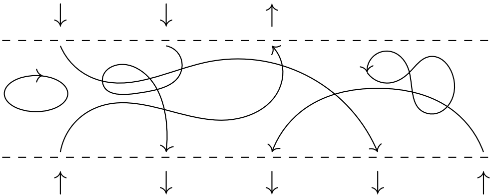
$$\psi_{n,2k}(\tau_{2r})(m_i \otimes v_a) = (1 - \delta_{a_r, a_{r+1}})(a_r, a_{r+1})(m_i \otimes v_{a_1} \otimes \cdots \otimes v_{a_{r-1}}) \otimes v_{a_r} \otimes \cdots \otimes v_{a_k}$$

and similarly for x_{2r+1} and τ_{2r+1}.

- Also z_l acts on the M component by a central element of $\mathbb{C}S_n$.

Heisenberg Category (1/4)

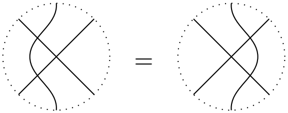
- The Heisenberg Category Heis is a monoidal category whose objects are generated by the two objects $\uparrow$ and $\downarrow$
- The morphism spaces are generated by certain diagrams modulo local relations. An example morphism $\alpha : \uparrow\uparrow\downarrow \to \uparrow\downarrow\downarrow\downarrow\uparrow$ is given by



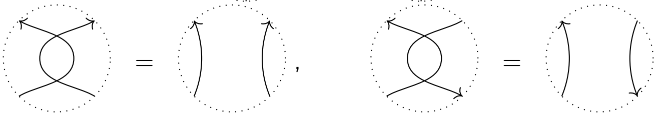
Heisenberg Category (2/4)

- Such diagrams are subject to the following local relations:

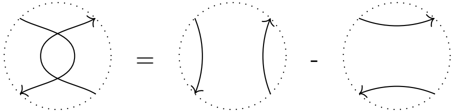
(H1)



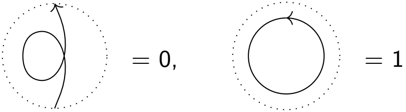
(H2)



(H3)

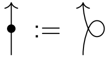


(H4)

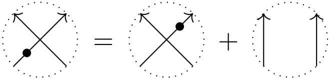


Heisenberg Category (3/4)

- We use a decoration to denote right curls:



- It turns out that these right curls satisfy the relation



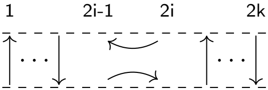
- With some more work it was proved that

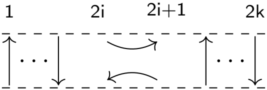
$$\mathrm{End}_{\mathrm{Heis}}(\uparrow^{\otimes k}) \cong \mathbb{C}[c_0, c_1, \ldots] \otimes \mathcal{H}_k,$$

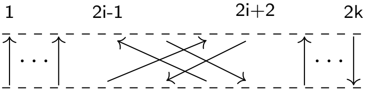
where c_i are commuting variables and are identified with a clockwise bubble with i decorations.

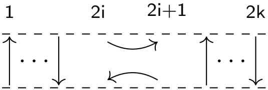
Heisenberg Category (4/4)

- It was shown by Likeng and Savage that the partition algebra $\mathcal{A}_{2k}$ can be embedded within $\mathrm{End}_{\mathrm{Heis}}((\uparrow \otimes \downarrow)^{\otimes k})$ via

$e_{2i-1} \mapsto$ 

, $e_{2i} \mapsto$ 

$s_i \mapsto$ 

$+$ 

- Noticing this Brundan and Vargas defined an affine partition category APar as a subcateogy of Heis, from which an alternative construction of an affine partition algebra is obtained as the endomorphism algebra $AP_k := \mathrm{End}_{\mathrm{APar}}((\uparrow \otimes \downarrow)^{\otimes k})$.

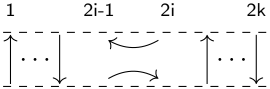
Surjection $\mathcal{A}_{2k}^{\text{aff}} \to \text{End}_{\text{Heis}}((\uparrow \otimes \downarrow)^{\otimes k})$ (1/2)

- We have a surjective algebra homomorphism

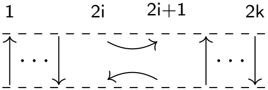
$$\phi_{2k} : \mathcal{A}_{2k}^{\text{aff}} \to \text{End}_{\text{Heis}}((\uparrow \otimes \downarrow)^{\otimes k})$$

given on the generators by

$$\phi_{2k}(e_{2i-1}) =$$

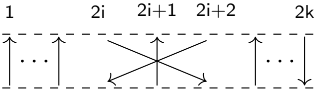


$$\phi_{2k}(e_{2i}) =$$

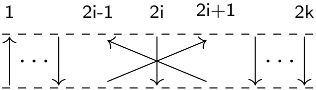


Surjection $\mathcal{A}_{2k}^{\text{aff}} \to \text{End}_{\text{Heis}}((\uparrow \otimes \downarrow)^{\otimes k})$ (2/2)

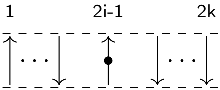
$\phi_{2k}(\tau_{2i+1}) =$



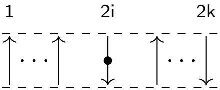
$\phi_{2k}(\tau_{2i}) =$



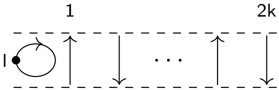
$\phi_{2k}(x_{2i-1}) =$



, $\phi_{2k}(x_{2i}) =$



$\phi_{2k}(z_l) =$



Additional Results

- The diagrammatic results we developed to prove the surjectivity of ϕ_{2k} can be generalised to prove the following:
  - The category APar the full submonoidal category of Heis generated by the object $\uparrow \otimes \downarrow$.
  - The algebra AP_k equals the endomorphism algebra $\mathrm{End}_{\mathrm{Heis}}((\uparrow \otimes \downarrow)^k)$.
  - We have a surjective homomorphism $\mathcal{A}_{2k}^{\mathrm{aff}} \to AP_k$.
- We suspect that the algebras $\mathcal{A}_{2k}^{\mathrm{aff}}$ and AP_k are isomorphic, although this appears difficult to prove as we have not got a basis for $\mathcal{A}_{2k}^{\mathrm{aff}}$ yet.

Thanks for listening!