Plethysm and the partition algebra

Rowena Paget

Based on joint work with Chris Bowman arXiv: 1809.08128, and joint work with Chris Bowman & Mark Wildon (in progress).

University of Kent

Athena SWAN Silver Award

ECU GENDER CHARTER

1. What is plethysm?

2. The partition algebra

3. Translating plethysm to the partition algebra

The symmetric group acting on set partitions

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The resulting transitive permutation module for $\mathbb{C}S_{mn}$ is called the Foulkes module.

Foulkes module

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$$\{\{1,2,\ldots,m\},\{m+1,\ldots,2m\},\ldots,\{(n-1)m+1,\ldots,nm\}\}$$

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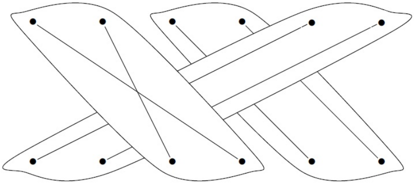

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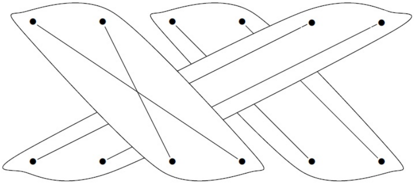

So the Foulkes module is just $1 \uparrow_{S_m \wr S_n}^{S_{mn}}$.

Foulkes' Conjecture

In 1950, H.O. Foulkes made the following conjecture:

If $m < n$ *then* $1\uparrow_{S_n \wr S_m}^{S_{mn}}$ *is a* $\mathbb{C}S_{mn}$*-submodule of* $1\uparrow_{S_m \wr S_n}^{S_{mn}}$.

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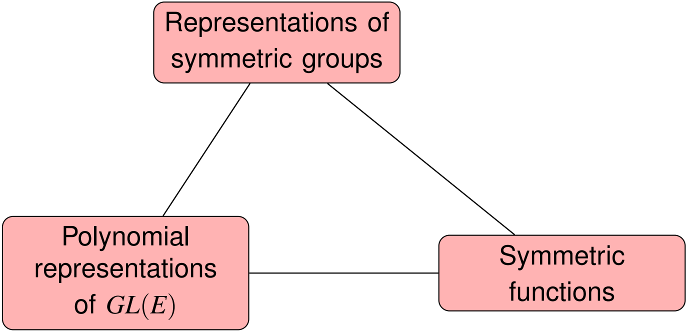
Foulkes' Conjecture holds for $m = 2$ (Thrall, 1942), $m = 3$ (Dent & Siemons, 2000), $m = 4$ (McKay, 2008), $m = 5$ (Cheung, Ikenmeyer & Mkrtchyan, 2016) and for $n >> m$ (Brion, 1993).

The general setting



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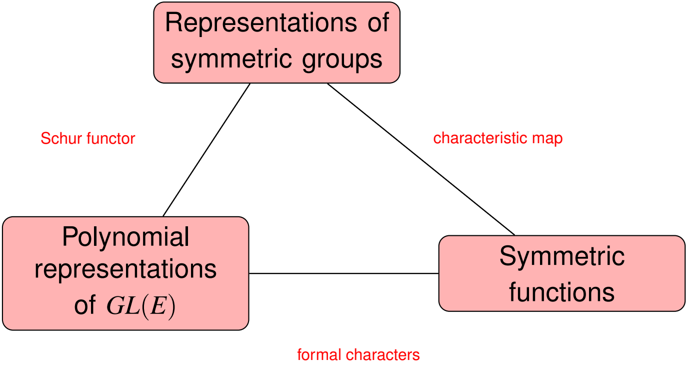
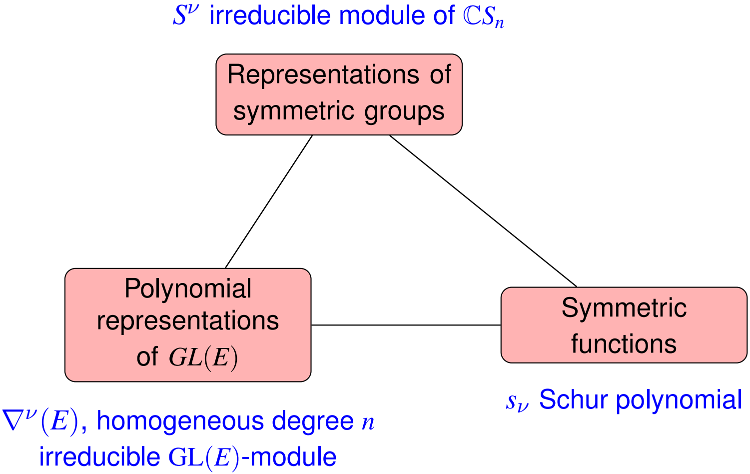


The general setting

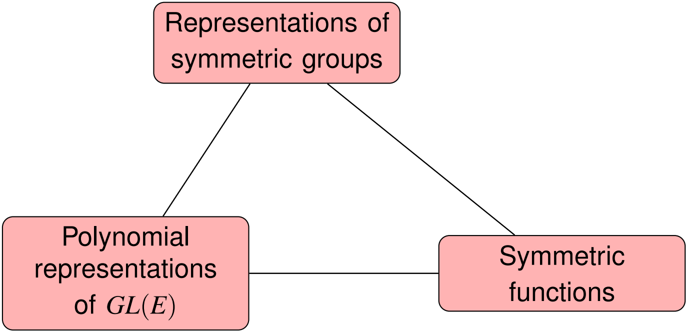
Let ν be a partition of $n \leq \dim(E)$.



The general setting

Special cases

$$S^{(n)} = \mathbb{C}$$
$$S^{(1,1,\dots,1)} = \mathrm{sgn}$$

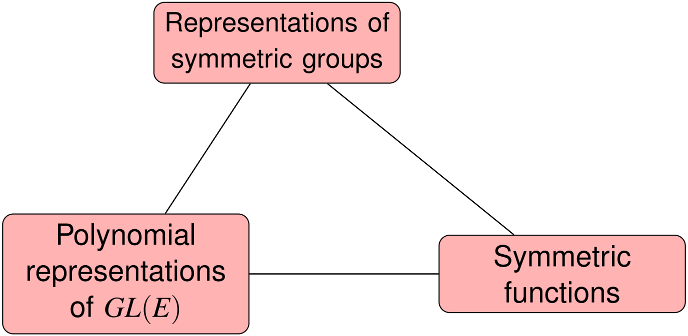


$$\nabla^{(n)}(E) = \mathrm{Sym}^n E$$
$$\nabla^{(1,1,\dots,1)}(E) = \bigwedge^n E$$

$$s_{(n)} = h_n \ ,$$
$$s_{(1,1,,\dots,1)} = e_n$$

Foulkes module and its analogues

$1\uparrow_{S_m \wr S_n}^{S_{mn}}$



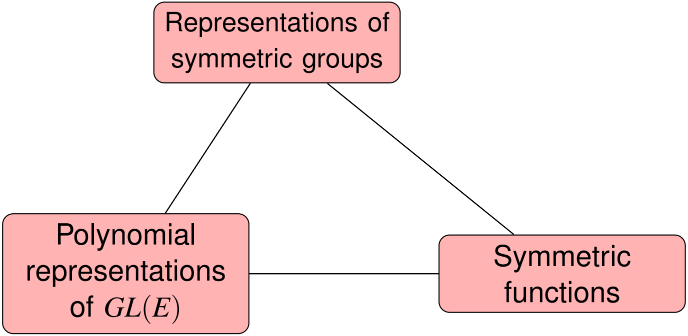
$\mathrm{Sym}^n(\mathrm{Sym}^m(E))$

$s_{(n)} \circ s_{(m)} = h_n \circ h_m$

Foulkes' Conjecture

Let $m < n$.

$1\uparrow_{S_n \wr S_m}^{S_{mn}} \hookrightarrow 1\uparrow_{S_m \wr S_n}^{S_{mn}}$



$\mathrm{Sym}^m(\mathrm{Sym}^n(E)) \hookrightarrow \mathrm{Sym}^n(\mathrm{Sym}^m(E))$

$s_n \circ s_m - s_m \circ s_n$

is positive sum of Schur polynomials

Littlewood's plethysm multiplication

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E.g., $s_2(x,y) = x^2 + xy + y^2$, so

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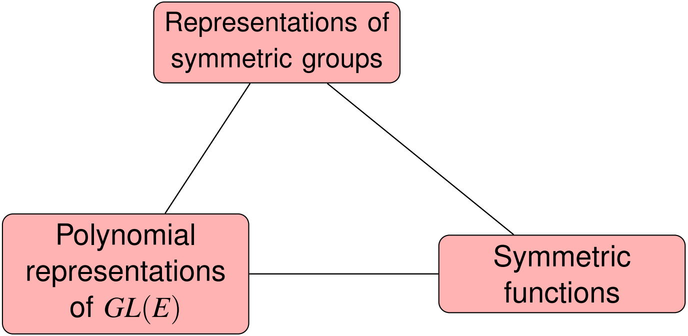
$$= s_4(x, y) + s_{(2,2)}(x, y).$$

Plethysm in general

Let ν be a partition of n and μ a partition of m.

$((S^{\mu})^{\otimes n} \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n}(S^{\nu}))\uparrow_{S_m \wr S_n}^{S_{mn}}$ generalised Foulkes module



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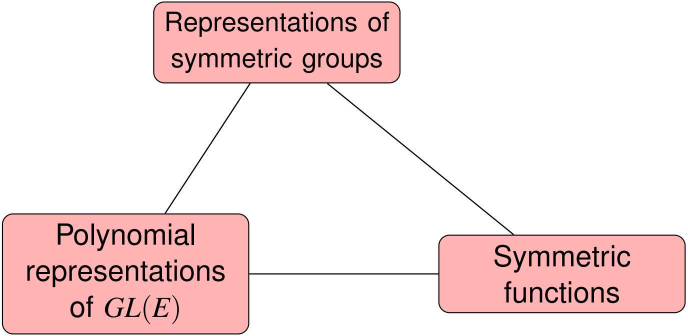

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$s_\nu \circ s_\mu$ plethysm

How does $s_\nu \circ s_\mu$ decompose? What are the plethysm coefficients $\langle s_\nu \circ s_\mu \, , \, s_\lambda \rangle$?

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$$\begin{array}{ccccc} \mathbb{C}\mathrm{GL}_{mn}(\mathbb{C}) & \circlearrowright & (\mathbb{C}^{mn})^{\otimes r} & \circlearrowleft & \mathbb{C}S_r \\ \cup & & & & \cap \\ \mathbb{C}S_{mn} & & & & ? \end{array}$$

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Answer (Martin 1991, Jones 1993): the partition algebra $P_r(mn)$.

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The partition algebra $P_r(mn)$ has basis all set partitions of the set $\{1, 2, \ldots, r, \overline{1}, \overline{2}, \ldots, \overline{r}\}$.

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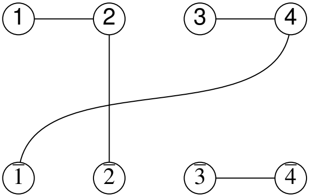
$x =$ 

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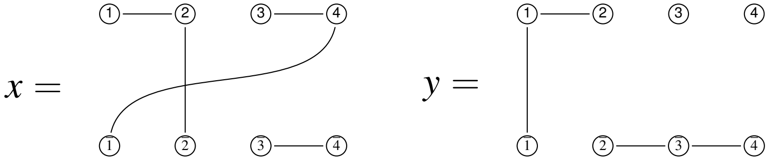

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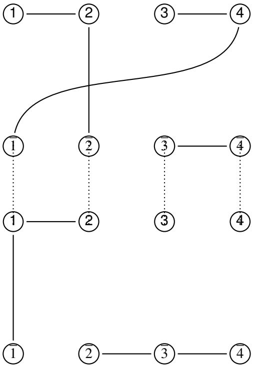

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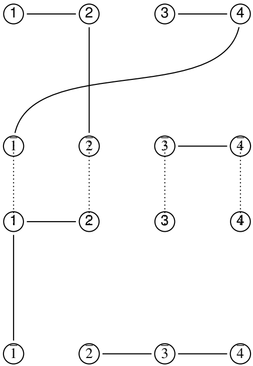
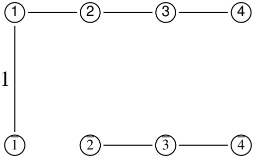


$xy = (mn)^1$ 

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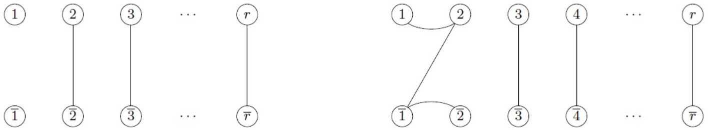

- Partition algebra $P_r(mn)$ is actually built up from the group algebras $\mathbb{C}S_r, \mathbb{C}S_{r-1}, \ldots, \mathbb{C}S_2, \mathbb{C}S_1, \mathbb{C}S_0 = \mathbb{C}$. It's a cellular algebra.

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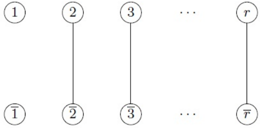

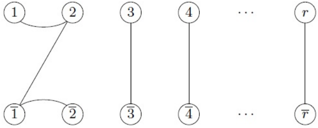

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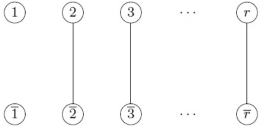

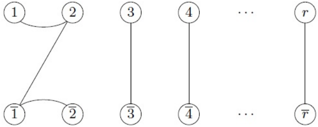

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- $P_r(mn)$ is semisimple if and only if $mn \notin \{0, 1, 2, \dots, 2r-2\}$.

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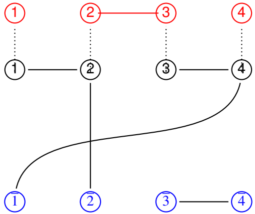

§3 Translating Plethysm to the Partition Algebra

We are interested in the plethysm coefficients

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$$\begin{array}{ccc} \mathbb{C}S_{mn} - \mathrm{mod} & \longrightarrow & \mathrm{mod} - P_r(mn) \\ 1\uparrow_{S_m \wr S_n}^{S_{mn}} & \mapsto & X \end{array}$$

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We are interested in the plethysm coefficients

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Schur-Weyl duality:

$$\begin{array}{ccccc} \mathbb{C}\mathrm{GL}_{mn}(\mathbb{C}) & \circlearrowright & (\mathbb{C}^{mn})^{\otimes r} & \circlearrowleft & \mathbb{C}S_r \\ \cup & & & & \cap \\ \mathbb{C}S_{mn} & & & & P_r(mn) \end{array}$$

$$\begin{array}{ccc} \mathbb{C}S_{mn} - \mathrm{mod} & \longrightarrow & \mathrm{mod} - P_r(mn) \\ 1\uparrow_{S_m \wr S_n}^{S_{mn}} & \mapsto & X \\ S^{(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k)} & \mapsto & L(\lambda_2, \lambda_3, \ldots, \lambda_k) \end{array}$$

(provided $\lambda_2 + \cdots \lambda_k \leq r$; otherwise it is killed.)
Our plethysm coefficient becomes

$$[1\uparrow_{S_m \wr S_n}^{S_{mn}} : S^\lambda]_{\mathbb{C}S_{mn}} = [X : L(\lambda_2, \lambda_3, \ldots, \lambda_k)]_{P_r(mn)}.$$

The diagrammatic Foulkes module

Define the diagrammatic Foulkes module to be the complex vector space F^r with basis the set of all (Λ, Λ') where Λ and Λ' are set partitions of $\{1, \ldots, r\}$ and Λ' is coarser than Λ. We draw the basis elements as diagrams and talk about Λ as the inner partition and Λ' as the outer partition.

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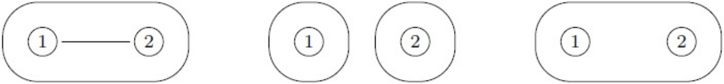

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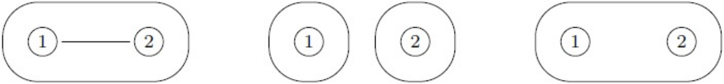

The $P_r(mn)$-module action: for x a partition algebra diagram we let

$$(\Lambda, \Lambda').x = m^a n^b (\Gamma, \Gamma')$$

if $\Lambda.x = m^a \Gamma$ in the natural $P_r(m)$-action on set partitions and $\Lambda'.x = n^b \Gamma'$ in the natural $P_r(n)$-action on set partitions.

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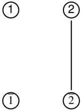
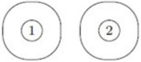
$(\Lambda, \Lambda') =$  $x =$  $(\Lambda, \Lambda').x = m$ 

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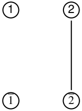
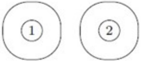
$(\Lambda, \Lambda') =$  $x =$  $(\Lambda, \Lambda').x = m$ 

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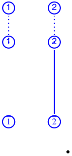

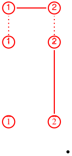

$\{\{1\},\{2\}\}.x = m\{\{1\},\{2\}\}$ $\{\{1,2\}\}.x = \{\{1\},\{2\}\}$

The correspondence

Proposition 1

Under the Schur functor $\mathrm{Hom}_{\mathbb{C}S_{mn}}(-, (\mathbb{C}^{mn})^{\otimes r})$,

$$\begin{array}{rcl} \mathbb{C}S_{mn} - \mathrm{mod} & \longrightarrow & \mathrm{mod} - P_r(mn) \\ 1\uparrow_{S_m \wr S_n}^{S_{mn}} & \mapsto & \textit{a quotient of } F^r. \end{array}$$

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Consequence: if $m, n \geq r$ and $\sum_{i \geq 2} \lambda_i \leq r$ then

$$[1\uparrow_{S_m \wr S_n}^{S_{mn}} : S^\lambda]_{\mathbb{C}S_{mn}} = [F^r : L(\lambda_2, \lambda_3, \ldots, \lambda_k)]_{P_r(mn)}.$$

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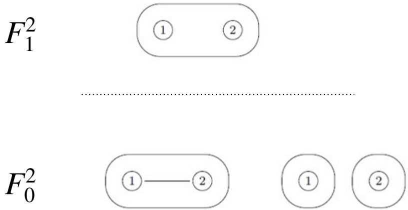

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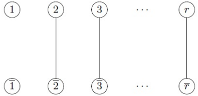
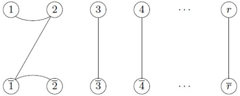
To see F^r_ℓ is a submodule we look at the effect of the partition algebra generators: the symmetric group generators behave well so it is just the two additional partition algebra generators to consider.





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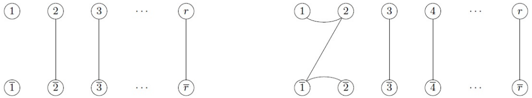

The $P_r(mn)$-action on the layer $F_\ell^r / F_{\ell-1}^r$ only depends on mn and not on the individual values of m and n.

Consequences for Foulkes' Conjecture

Theorem 2

Suppose $m, n \geq \sum_{i\geq 2} \lambda_i$ *then* $[1\uparrow_{S_m \wr S_n}^{S_{mn}} : S^\lambda] = [1\uparrow_{S_n \wr S_m}^{S_{mn}} : S^\lambda]$.

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A geometric proof of this was given in 1998 by Manivel.

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Theorem 3

The double-sequence $\langle s_{(n)} \circ s_{(m)}, s_{\lambda_{[mn]}} \rangle$ stabilises and the stable values are achieved whenever $m, n \geq |\lambda|$.

The stability for n was originally proved by Brion, and for m by Carré-Thibon.

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