(Some) Gram Determinants for A_n nets

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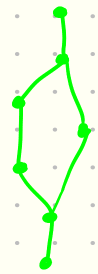
§ Cellular Categories and motivations



Def A cellular algebra A over R

- a set of "cell indices" Λ partially ordered
- sets of tableaux $M(\lambda)$ for $\lambda \in \Lambda$
- elements C^{λ}_{AB} for A,B tableaux in $M(\lambda)$
- an involution ι sending $C^{\lambda}_{AB} \longrightarrow C^{\lambda}_{BA}$

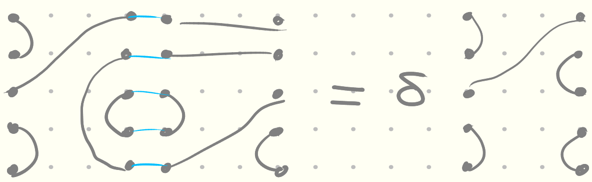
satisfying

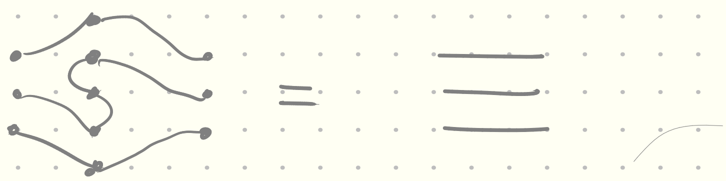
i) $\{ C^{\lambda}_{AB} \}$ is an R-basis

ii) $C^{\lambda}_{AB} \cdot C^{\lambda}_{CD} \in A^{\leq \lambda}$

Cellular category $\qquad C^{\lambda}_{AB} = C_A \circ C_B$

TL_n





Inner product on linear span of $M(\lambda)$.

Radical of this is the max submodule of $M(\lambda)$. $\longleftarrow$

If all cell modules irreducible (i.e. all inner products non-degenerate)
$\Longrightarrow$ semi simple

Hence care about determinants of Gram matrix

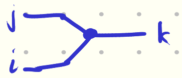
§2 Nets, light ladders, clasps

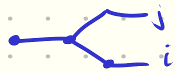
$U_q(\mathfrak{sl}_n) \longrightarrow \mathfrak{sl}_n$ $\qquad [n] \longrightarrow n$ $\qquad q \longrightarrow 1$, $\delta \longrightarrow 2$

$n-1$ fundamental reps $\qquad V_i = \{\Lambda^i \mathbb{C}^n\}_{i=1}^{n-1}$

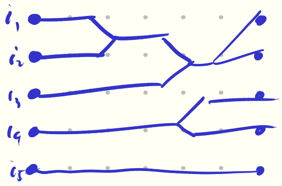
$V_0 = V_n = \mathbb{C}$

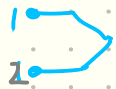
$i+j=k \qquad V_i \otimes V_j \longrightarrow V_k \qquad\qquad V_k \longrightarrow V_i \otimes V_j$



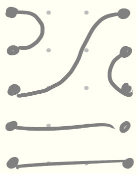


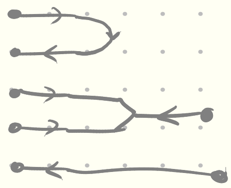
$V_{i_1} \otimes V_{i_2} \otimes \cdots V_{i_k}$





$\mathfrak{sl}_2$ $\qquad$ TL $\qquad$ $\mathfrak{sl}_3$





$\Lambda \qquad$ object $\underline{x} \in \{1, \dots, n-1\}^*$

$w(\underline{x}) = (x_1, \dots, x_{n-1}) > (x_1, \dots, x_{i-1}+1, x_i - 2, x_{i+1}+1, \dots, x_{n-1})$

$\underline{x} > y$

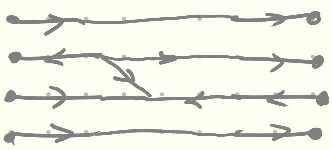
Def A compatible family for weight λ is a set of maps

$\varphi_{xy} : X \to Y \qquad$ for all X, Y of weight λ

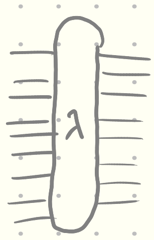
$\varphi_{xx} = \mathbb{1}_x \qquad$ modulo things $< \lambda$

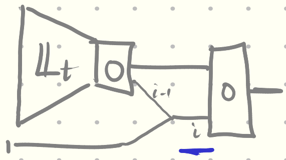
$\varphi_{xy} \circ \varphi_{zx} = \varphi_{zy} \qquad$ ——— " ———

Neutral ladders



Clasps





§3 Gram determinants

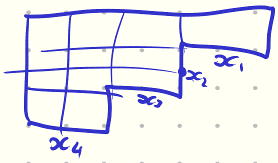
$$V_1 \otimes V_1 \otimes V_1 \dots V_1 \longrightarrow$$

$$\underline{1}^a \longrightarrow \underline{x}$$

$(\underline{1}^a, \underline{x})$ $\quad \lambda = (\lambda_1, \dots, \lambda_n) \vdash a$

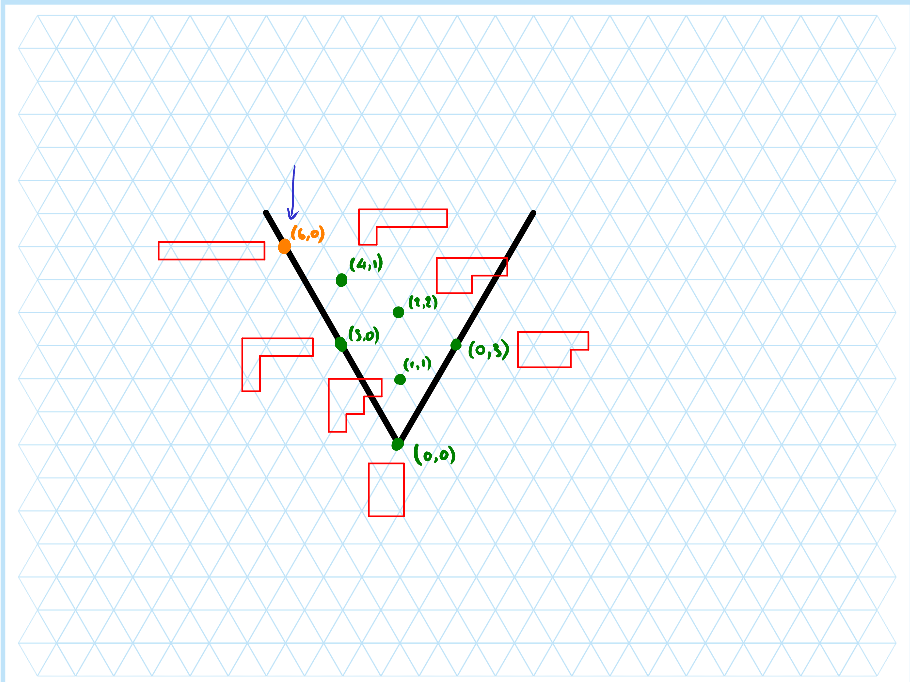
$$\lambda_i = \frac{a - \sum_{j=1}^{i-1} j x_j + \sum_{j=i}^{n-1} (n-j) x_j}{n}$$

$$(\lambda_1, \dots, \lambda_n) \succ (\lambda_1, \dots, \lambda_i + 1, \lambda_{i+1} - 1, \dots, \lambda_n)$$



$V_1^{\otimes 6}$

Weight	Partition
$(6,0)$	(6)
$(4,1)$	$(5,1)$
$(2,2)$	$(4,2)$
$(0,3)$	$(3,3)$
$(3,0)$	$(4,1,1)$
$(1,1)$	$(3,2,1)$
$(0,0)$	$(2,2,2)$

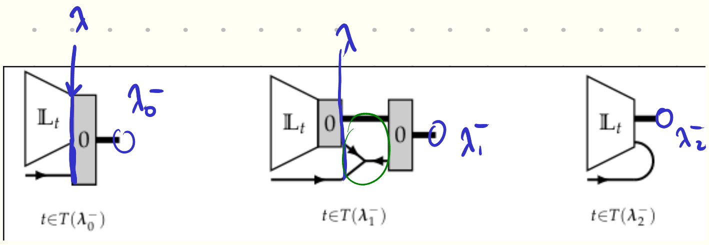


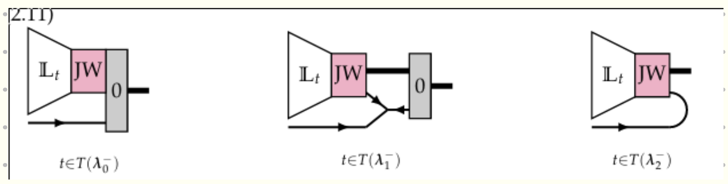
There is a bijection between standard tableaux of shape λ and a basis of

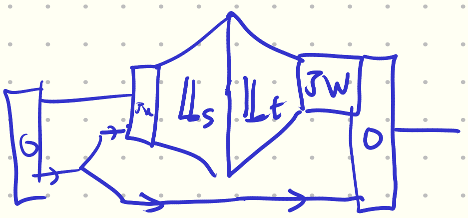
$$\mathrm{Hom}^{<\lambda}(V_1^{\otimes a}, V_{x^1} \otimes V_{x^2} \otimes \cdots V_{x^k})$$

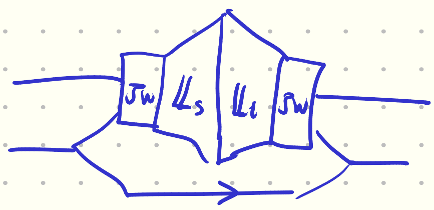
$$= S(\lambda)$$

$$\lambda_i^- = (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_n)$$

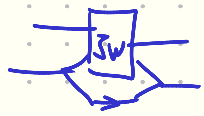




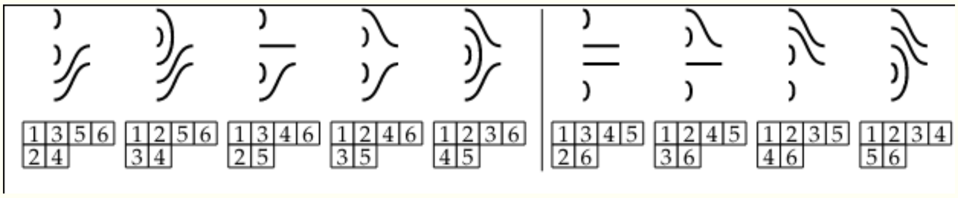




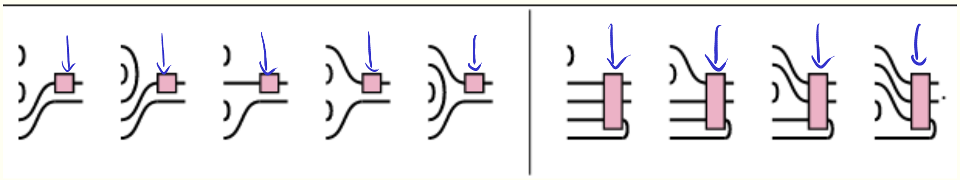
$\langle \mathbb{L}_s, \mathbb{L}_t \rangle$



$a=6 \quad n=2$

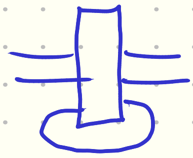


$$
G(6,2) = \begin{pmatrix}
\delta^2 & \delta & \delta & 1 & \delta & 0 & 0 & 0 & 0 \\
\delta & \delta^2 & 1 & \delta & 1 & 0 & 0 & 0 & 0 \\
\delta & 1 & \delta^2 & \delta & 1 & \delta & 1 & 0 & 0 \\
1 & \delta & \delta & \delta^2 & \delta & 1 & \delta & 1 & \delta \\
\delta & 1 & 1 & \delta & \delta^2 & 0 & 1 & \delta & 1 \\
0 & 0 & \delta & 1 & 0 & \delta^2 & \delta & 0 & 0 \\
0 & 0 & 1 & \delta & 1 & \delta & \delta^2 & \delta & 1 \\
0 & 0 & 0 & 1 & \delta & 0 & \delta & \delta^2 & \delta \\
0 & 1 & 0 & \delta & 1 & 0 & 1 & \delta & \delta^2
\end{pmatrix}
$$

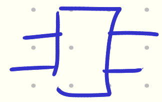


$$
G'(6,2) = \begin{pmatrix}
\delta^2 & \delta & \delta & 1 & \delta & 0 & 0 & 0 & 0 \\
\delta & \delta^2 & 1 & \delta & 1 & 0 & 0 & 0 & 0 \\
\delta & 1 & \delta^2 & \delta & 1 & 0 & 0 & 0 & 0 \\
1 & \delta & \delta & \delta^2 & \delta & 0 & 0 & 0 & 0 \\
\delta & 1 & 1 & \delta & \delta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\delta^4-2\delta^2}{\delta^2-1} & \frac{\delta^3-2\delta}{\delta^2-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\delta^3-2\delta}{\delta^2-1} & \frac{\delta^4-2\delta^2}{\delta^2-1} & \frac{\delta^3-2\delta}{\delta^2-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta^3-2\delta}{\delta^2-1} & \frac{\delta^4-2\delta^2}{\delta^2-1} & \frac{\delta^3-2\delta}{\delta^2-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta^3-2\delta}{\delta^2-1} & \frac{\delta^4-2\delta^2}{\delta^2-1}
\end{pmatrix}
$$

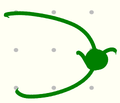
$$
= \left(\begin{array}{c|c} G(5,1) & 0 \\ \hline 0 & \frac{\delta^3-2\delta}{\delta^2-1} G(5,3) \end{array}\right).
$$

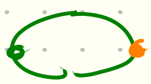


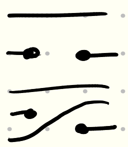
$$= \frac{\delta^3-2\delta}{\delta^2-1}$$

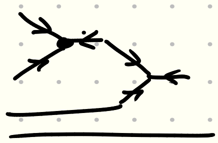


§4 Graphs and unravelling



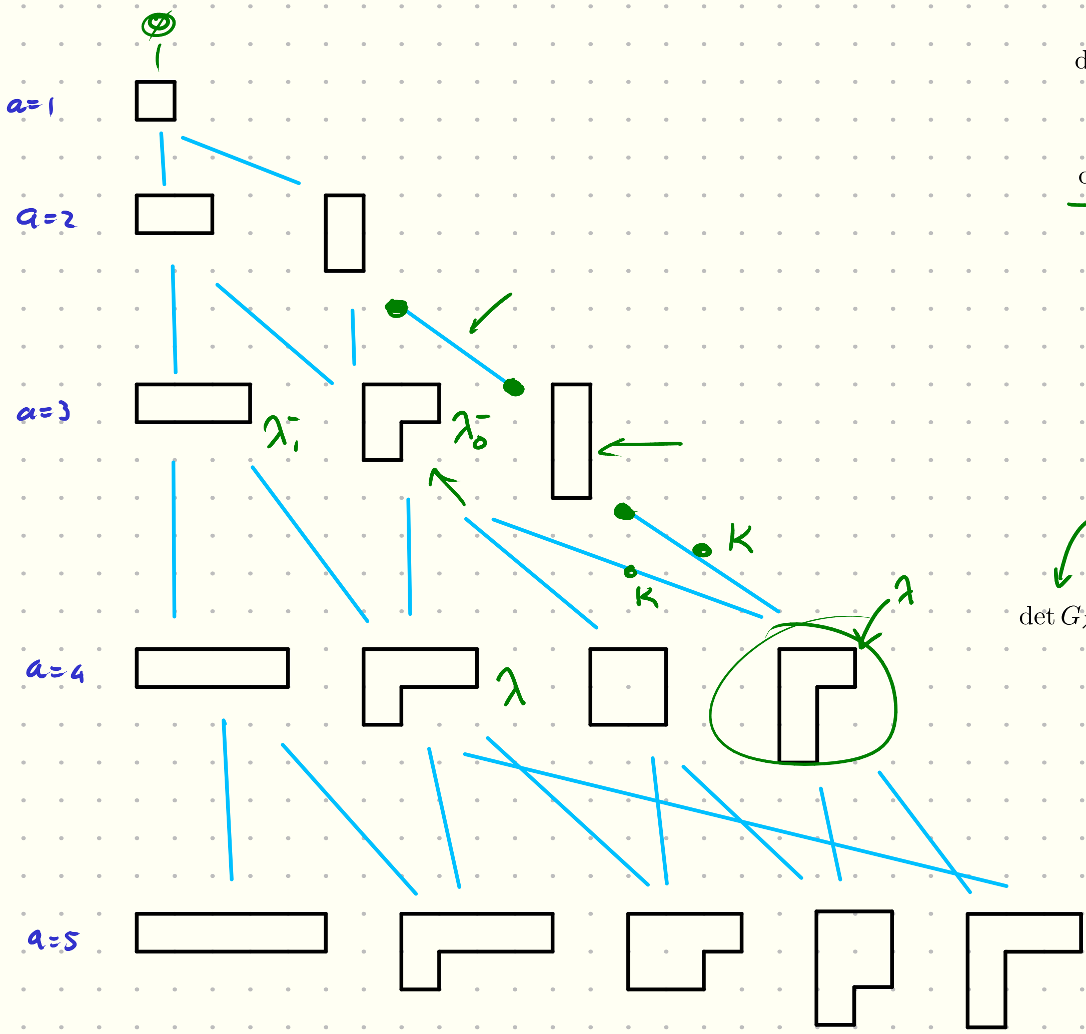






$$\det G_\lambda = \prod_{t \in T(\lambda)} \prod_{e \in t : \mu_1 \to \mu_2} \kappa_{i(e)}(\mu_1)^{d_{\mu_1}}$$

$$\det G_\lambda = \prod_{e \in \mathbb{Y}_n : \mu_1 \to \mu_2} \kappa_{i(e)}(\mu_1)^{d_{\mu_1} d_{\lambda \setminus \mu_2}}$$



$\in \mathbb{Z}[q+q^{-1}]$

$$\det G_\lambda = \prod_{i=1}^{n} \left(\det G_{\lambda_i^-}\right) \kappa_i(\lambda_i^-)^{d_{\lambda_i^-}}$$

$V_k^{\otimes n}$

n=2, n=3 ✓

Conjecture (Elias): Let λ be dominant and $\mu \in \Omega(a)$. We conjecture that whenever $\lambda + \mu$ is dominant,

$$\kappa_{\lambda,\mu} = \prod_{\alpha \in \Phi(\mu)} \frac{[\langle \lambda + \rho, \alpha \rangle]}{[\langle \lambda + \rho, \alpha \rangle - 1]}$$

where $\Phi(\mu)$ is a subset of positive roots not explained here,

and $\Omega(a)$ is the set of weights of the fundamental representaiton indexed by a.