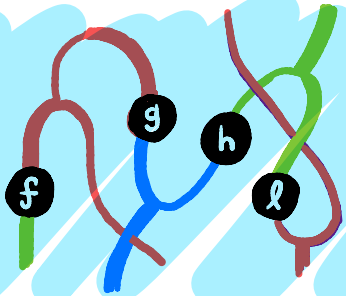
Superalgebra Deformations of Web Categories



Robert Muth
Duquesne University
Pittsburgh, PA USA

OIST Representation Theory Seminar
July 5, 2022

Jt. with Nick Davidson, Jonathan Kujawa & Jieru Zhu

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  - If $f \in A_\varepsilon$, we write $\bar{f} = \varepsilon$, and refer to ε as the parity of f.

  - We will always assume that A is free of finite rank $/ \mathbb{k}$.

Some examples

- $A = A_{\bar{0}} = \mathbb{K}$

SOME EXAMPLES

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- The Clifford algebra $A = \mathbb{k}[c]/(c^2-1)$.

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  <u>Modulo</u>:
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- The parity of a path q is given by $\bar{q} = \text{length}(q) \bmod 2$.
- We use Z_ℓ to indicate the (even) subalgebra of length 0 paths.

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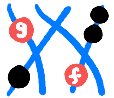
- Affine Hecke algebra $\mathcal{H}_n$ $\rightsquigarrow$ Affine wreath algebras $\mathcal{H}_n(A)$ 

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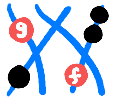
- Affine Hecke algebra $\mathcal{H}_n$ $\rightsquigarrow$ Affine wreath algebras $\mathcal{H}_n(A)$ 

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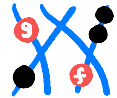
- Affine Hecke algebra $\mathcal{H}_n$ $\rightsquigarrow$ Affine wreath algebras $\mathcal{H}_n(A)$ 

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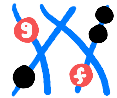
- Affine Hecke algebra $\mathcal{H}_n$ $\rightsquigarrow$ Affine wreath algebras $\mathcal{H}_n(A)$ 

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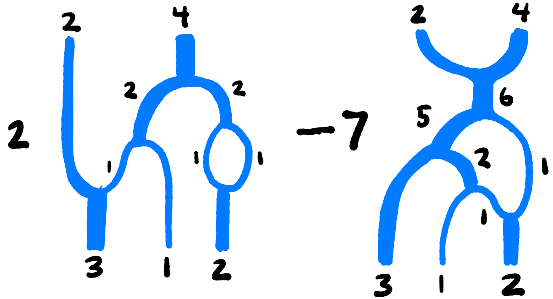
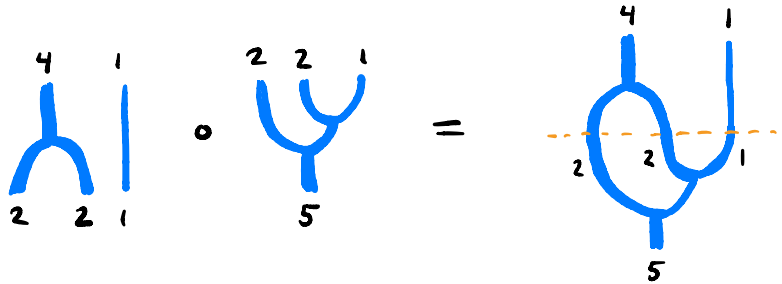


$: (3,1,2) \longrightarrow (2,4)$

Composition of morphisms is given by vertical concatenation:



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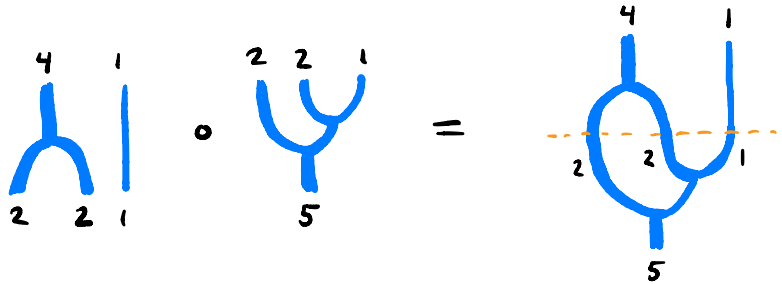
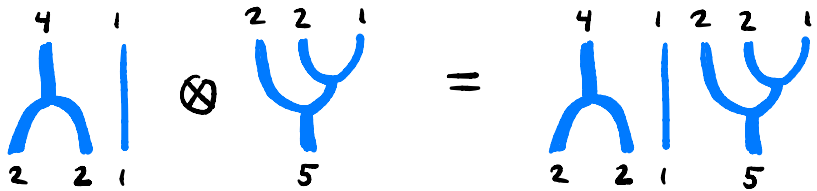


Monoidal structure given by horizontal concatenation:



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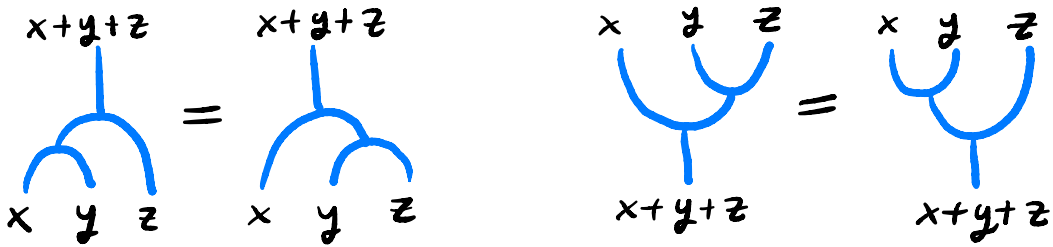

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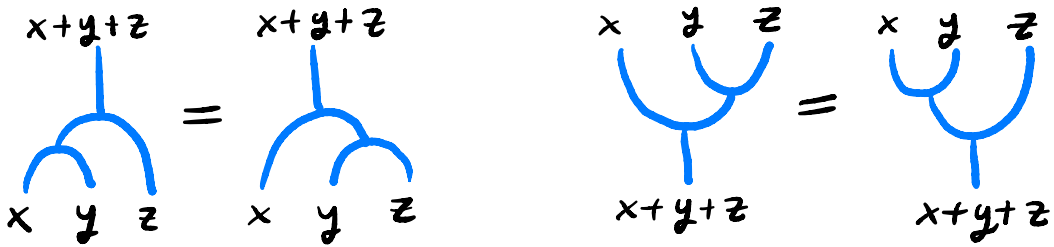

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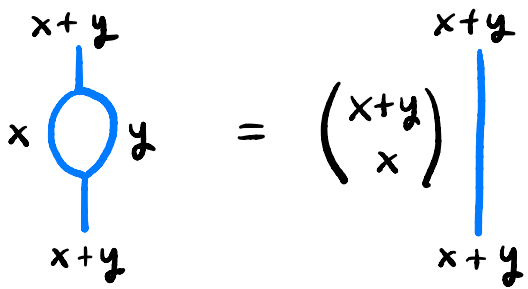

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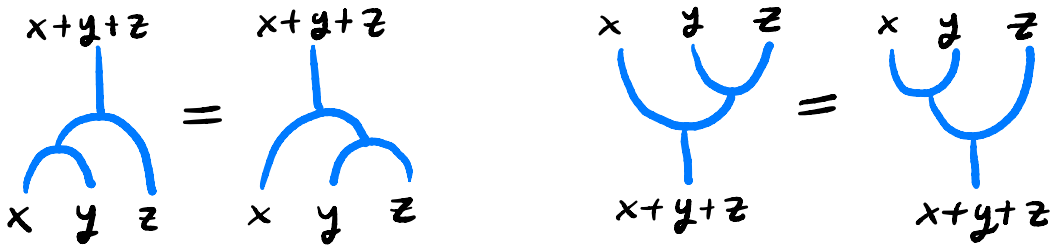

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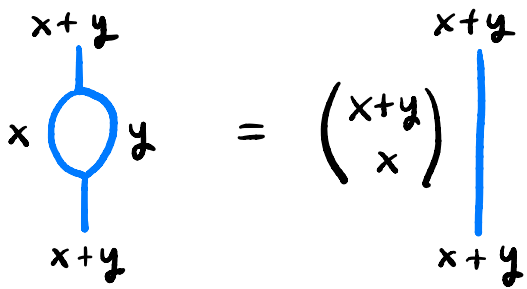

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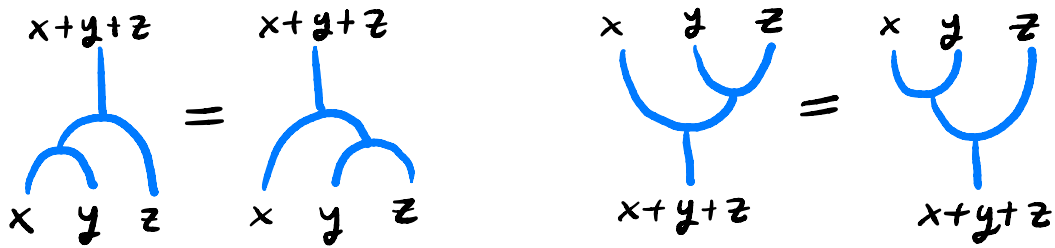

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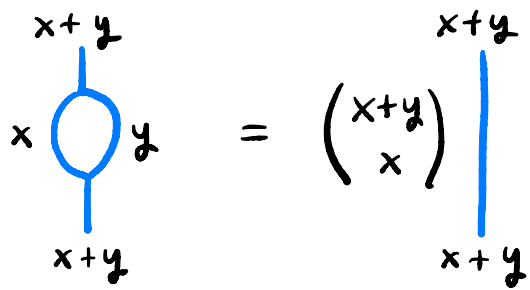

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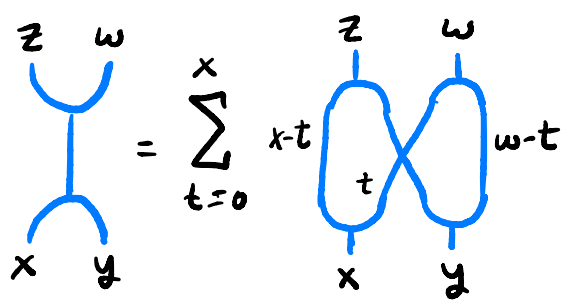

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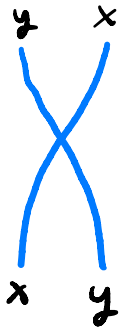

$$:= \sum_{t=0}^{x} (-1)^t \, t$$

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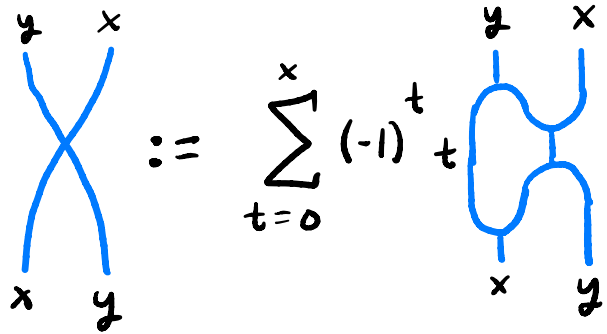

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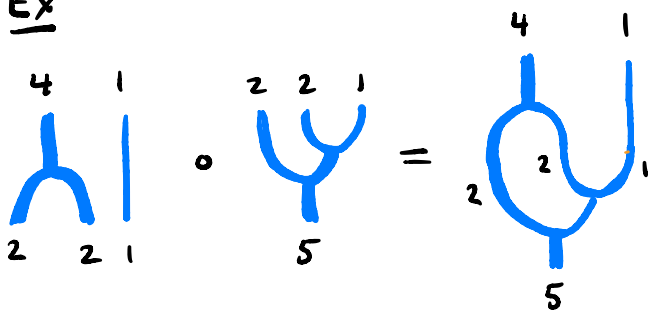

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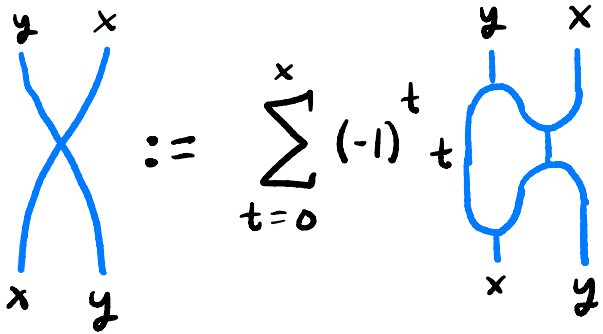

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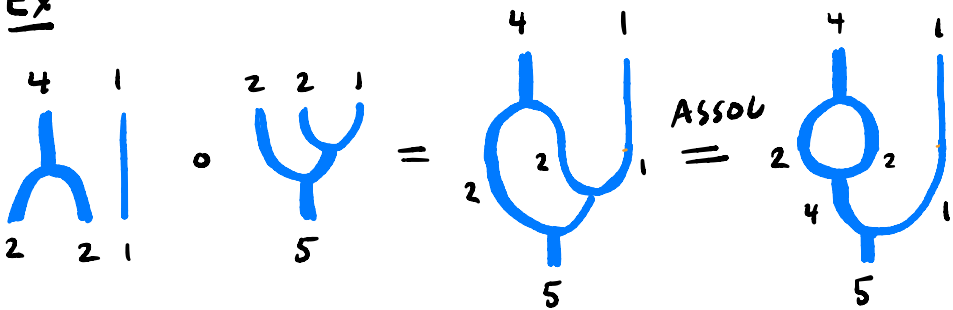

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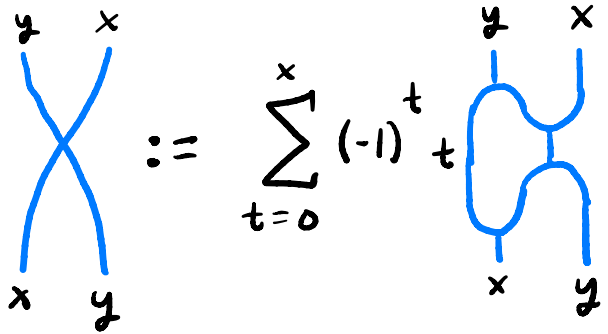

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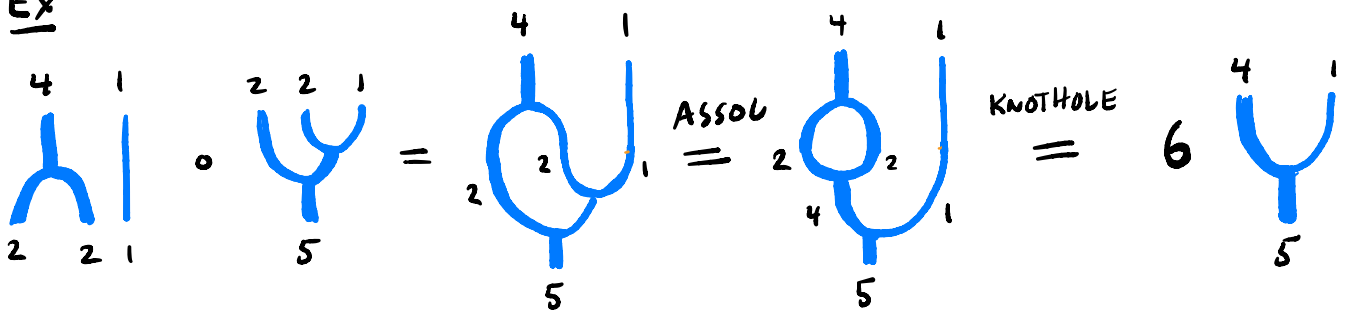

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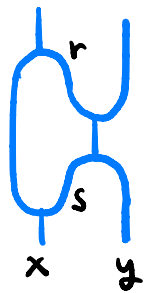
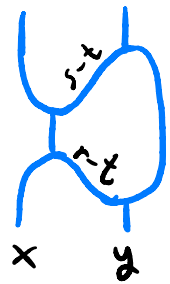


$$= \sum_{t=0}^{s} \binom{x-y+r-s}{t} (-1)^t$$



The category mod-$\mathfrak{gl}_n(\mathbb{k})_\mathcal{S}$

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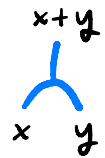

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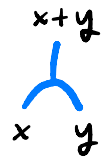

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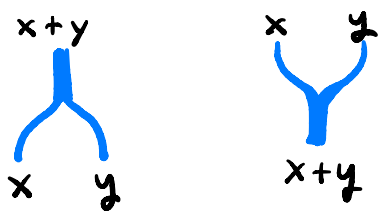

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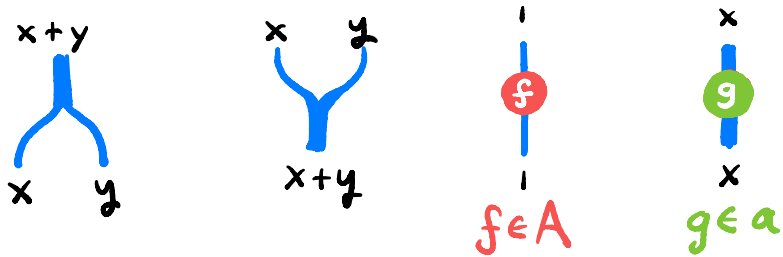

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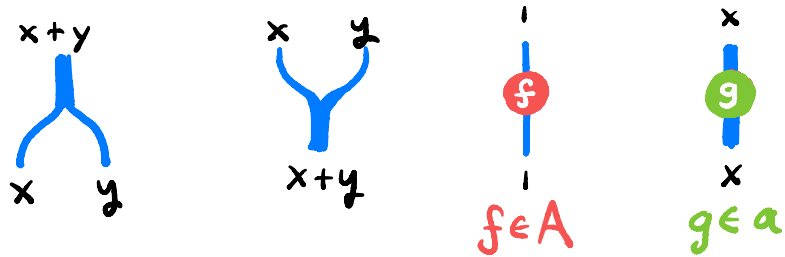
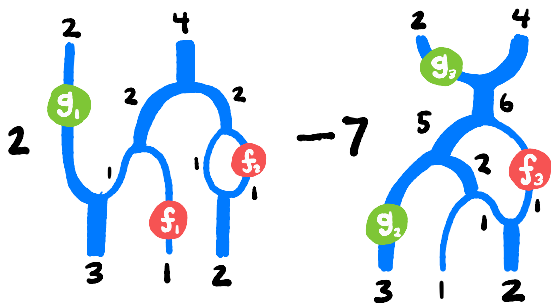


Ex



$: (3,1,2) \longrightarrow (2,4)$

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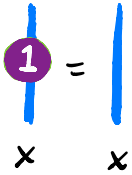

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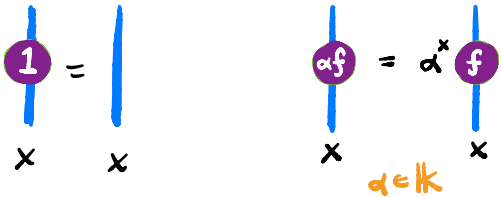

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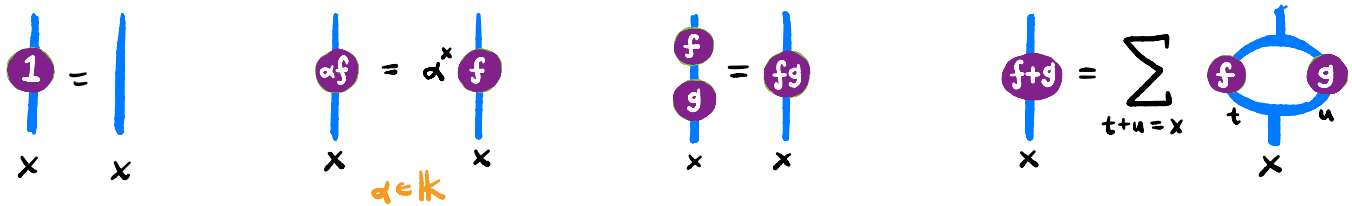

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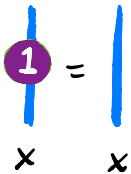

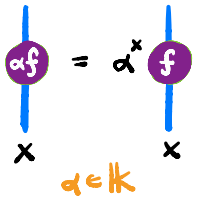

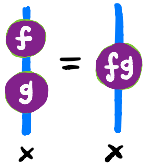

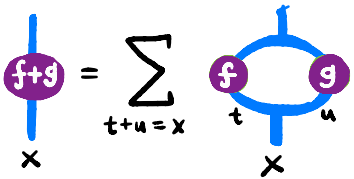

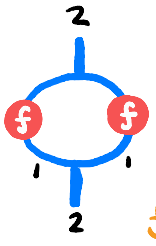

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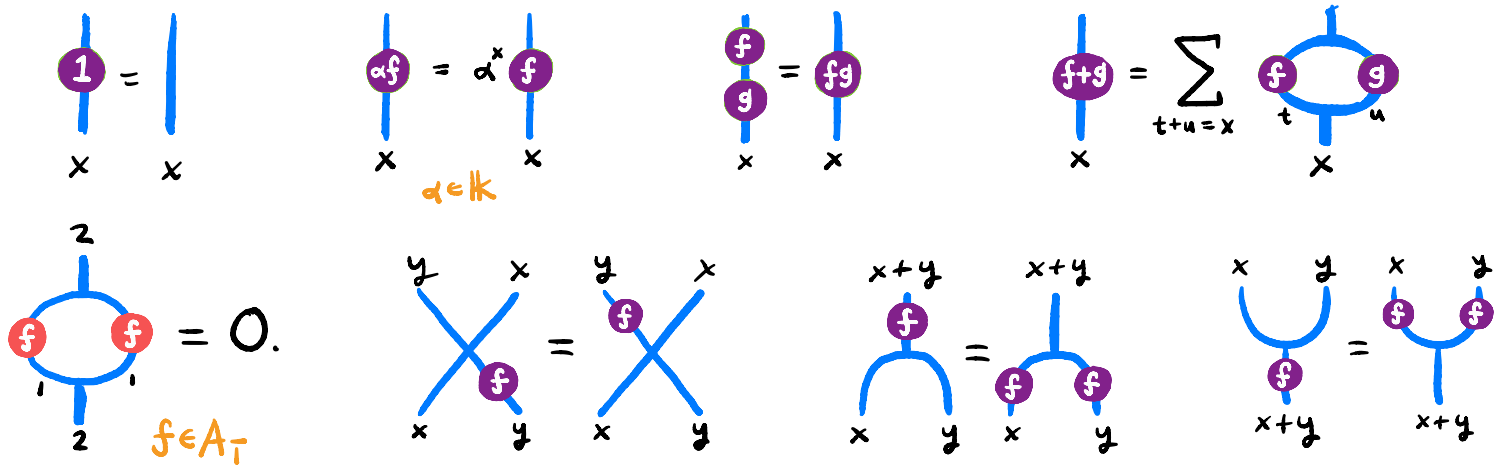

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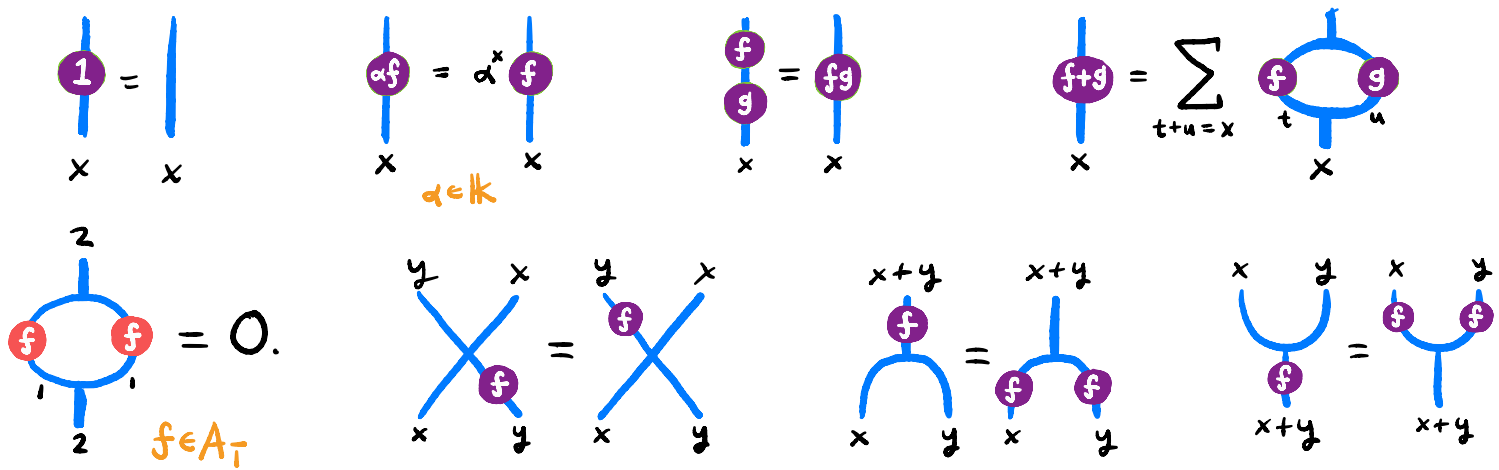

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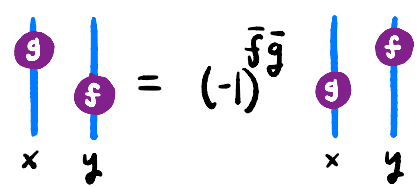

The Lie superalgebra $\mathfrak{gl}_n(A)$

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$$v_r^f = (0, \dots, f, \dots, 0)$$

$\uparrow$ rth component.

Symmetric power modules for $\mathfrak{gl}_n(A)$.

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- Left multiplication morphisms:

$$L_f^x : S^x V_n^A \longrightarrow S^x V_n^A$$
$$v_{r_1}^{g_1} \cdots v_{r_x}^{g_x} \longmapsto v_{r_1}^{fg_1} \cdots v_{r_x}^{fg_x}$$

for all $\begin{cases} f \in A, \ x = 1; \\ f \in a, \ x > 1. \end{cases}$

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$\longmapsto L_f^x$

- $F_n^{A,a}$ is essentially surjective
- $F_n^{A,a}$ is asymptotically faithful if $a = A_{\bar{0}}$ or $\mathrm{char}(\Bbbk) = 0$.
- This "defining representation" for $\mathrm{Web}_{\mathfrak{gl}}^{A,a}$ yields bases for morphism spaces in $\mathrm{Web}_{\mathfrak{gl}}^{A,a}$ in these cases.
- $F_n^{A,a}$ is full when Howe duality holds for $(\mathfrak{gl}_m(A), \mathfrak{gl}_n(A))$. e.g. $\Bbbk = \mathbb{C}$, A semisimple.

- For various choices of A, a, the category $\mathrm{Web}^{A,a}_{\mathfrak{gl}}$ generalizes some known web constructions:
  - $A = \Bbbk \rightsquigarrow \mathrm{Web}_{\mathfrak{gl}}$ (or "Schur category")

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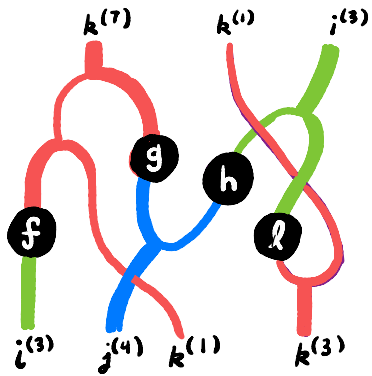
  - $A = Cl_1 \rightsquigarrow \mathrm{Web}_{\mathfrak{q}}$ (Brown-Kujawa-Davidson)
- More generally, we construct the category $\mathrm{Web}_{\mathfrak{gl}}^{\mathcal{A},a}$, associated to any small $\Bbbk$-linear SUPERCATEGORY $\mathcal{A}$, and even subcategory a.



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  - We have the star product $*$:

$$* : M_n(A)^{\otimes d} \otimes M_n(A)^{\otimes e} \longrightarrow M_n(A)^{\otimes d+e}$$

$$\left(E^{f_1}_{r_1 s_1} \otimes \cdots \otimes E^{f_d}_{r_d s_d}\right) * \left(E^{g_1}_{t_1 u_1} \otimes \cdots \otimes E^{g_e}_{t_e s_e}\right)$$

$$= \sum_{\sigma \in \mathfrak{S}_d \times \mathfrak{S}_e \backslash \mathfrak{S}_{d+e}} \left(E^{f_1}_{r_1 s_1} \otimes \cdots \otimes E^{f_d}_{r_d s_d} \otimes E^{g_1}_{t_1 u_1} \otimes \cdots \otimes E^{g_e}_{t_e s_e}\right)^{\sigma}$$

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  - And the shifted star product $\hat{*}$:

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$$\left(E^{f_1}_{r_1 s_1} \otimes \cdots \otimes E^{f_{d_1}}_{r_{d_1} s_{d_1}}\right) \hat{*} \left(E^{g_1}_{t_1 u_1} \otimes \cdots \otimes E^{g_{d_2}}_{t_{d_2} s_{d_2}}\right) = \left(E^{f_1}_{r_1 s_1} \otimes \cdots \otimes E^{f_{d_1}}_{r_{d_1} s_{d_1}}\right) * \left(E^{g_1}_{t_1+n_1, u_1+n_1} \otimes \cdots \otimes E^{g_{d_2}}_{t_{d_2}+n_1, s_{d_2}+n_1}\right)$$

The Schurification $T_a^A(n,d)$.

<u>Def</u> Set $S^A(n,d)$ to be the superalgebra of $\mathfrak{S}_d$-(super) invariants:

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- If $A_{\mathbb{F}}$ is SYMMETRIC/CELLULAR/QUASI-HEREDITARY, then $T_a^A(n,d)_{\mathbb{F}}$ will be as well*, but $S^{A_{\mathbb{F}}}(n,d)$ may not be.

* SOME RESTRICTIONS APPLY

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- Other conjectures: Schur algebra, Sergeev superalgebra, ...
- A presentation of $T_a^A(n,d)_{\mathbb{F}}$ would be useful.

$\mathrm{Web}^{A,a}_{\mathfrak{gl}}$ and Schurifications

- For every n-part composition $\underline{x} = (x_1, \dots, x_n)$ of d, there is an idempotent $e_{\underline{x}} = (E'_{11})^{\otimes x_1} * \cdots * (E'_{nn})^{\otimes x_n} \in T^A_a(n,d)$.

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- Composition of morphisms is given by multiplication.
- Monoidal structure on morphisms is given by $\hat{*}$.

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Theorem (Davidson–Kujawa–M.–Zhu).

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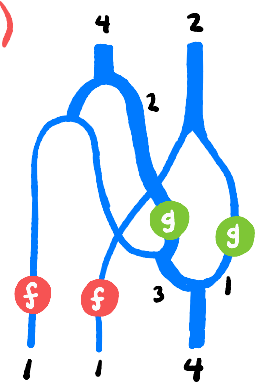

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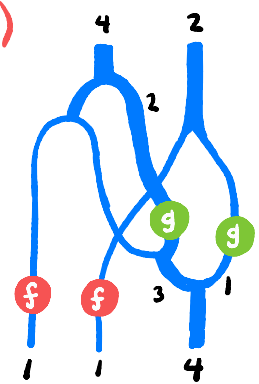

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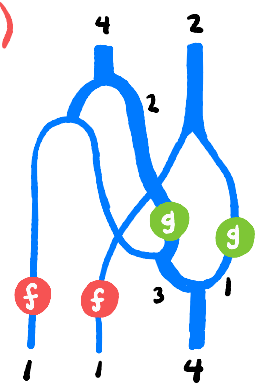

- $\mathrm{Web}^{A,a}_{\mathfrak{gl}}$ provides a diagrammatic pres. via local relations for Schurifications.
- Can be defined directly over $\mathbb{F}$; no need to work via integral form.

Howe duality and $\mathfrak{gl}_n(A)$.

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- For semisimple A, we have strong multiplicity-free decomposition:

$$S^d(V_m^A \underset{A}{\otimes} V_n^A) \cong \bigoplus_{\lambda} L_m(\lambda) \circledast L_n(\lambda)$$

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Howe duality and webs.

Theorem (DKMZ) There is a full, surjective functor:

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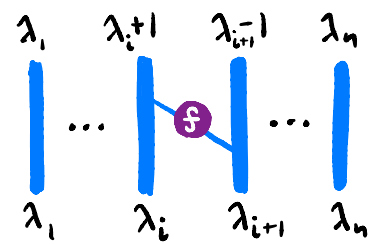

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SURJECTIVE UNDER HOWE DUALITY

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SURJECTIVE UNDER HOWE DUALITY

$$\dot{\mathbf{U}}(\mathfrak{gl}_m(A)) \longrightarrow \mathrm{End}_{\mathfrak{gl}_n(A)}\left(S^d\left(V_m^A \otimes_A V_n^A\right)\right)$$

$$\cong$$

$$\mathrm{End}_{\mathfrak{gl}_n(A)}\left(\bigoplus_{\underline{x} \vDash_m d} S^{x_1} V_n^A \otimes \cdots \otimes S^{x_m} V_n^A\right)$$

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$$\cong$$

$$\mathrm{Web}^{A,a}_{\mathfrak{gl},m} \xrightarrow{F_n^{A,a}} \mathrm{End}_{\mathfrak{gl}_n(A)}\left(\bigoplus_{\underline{x} \models_m d} S^{x_1} V_n^A \otimes \cdots \otimes S^{x_m} V_n^A\right)$$

DEFINING REPRESENTATION of $\mathrm{Web}^{A,a}_{\mathfrak{gl}}$.

Howe duality and webs.

SURJECTIVE UNDER HOWE DUALITY

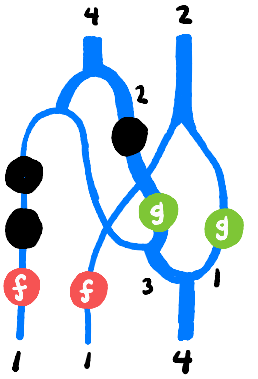
$$\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{gl}_m(A)) \arrow[r] \arrow[d, "H_n^{A,a}"'] \arrow[dr, phantom, "\circ"] & \mathrm{End}_{\mathfrak{gl}_n(A)}\left(S^d(V_m^A \otimes_A V_n^A)\right) \arrow[d, phantom, "\cong"] \\
\mathrm{Web}^{A,a}_{\mathfrak{gl},m} \arrow[r, "F_n^{A,a}"] & \mathrm{End}_{\mathfrak{gl}_n(A)}\left(\bigoplus_{\underline{x} \vDash_m d} S^{x_1}V_n^A \otimes \cdots \otimes S^{x_m}V_n^A\right)
\end{tikzcd}$$

DEFINING REPRESENTATION of $\mathrm{Web}^{A,a}_{\mathfrak{gl}}$.

Future/Ongoing

- Affine web categories for Frobenius A.

$$\mathrm{AffWeb}_{\mathfrak{gl}}^{A,a} \longrightarrow \mathrm{Fun}\left(\text{mod-}\mathfrak{gl}_n(A), \text{mod-}\mathfrak{gl}_n(A)\right)$$



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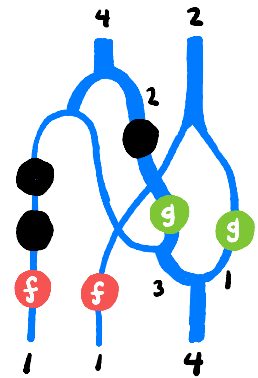
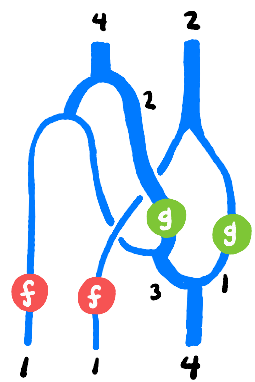


- Quantum web deformations

$$\mathrm{Web}^{A,a}_{\mathfrak{gl},q} \longrightarrow \text{mod-}U_q(\mathfrak{gl}_n(A))_S$$



THANK YOU!