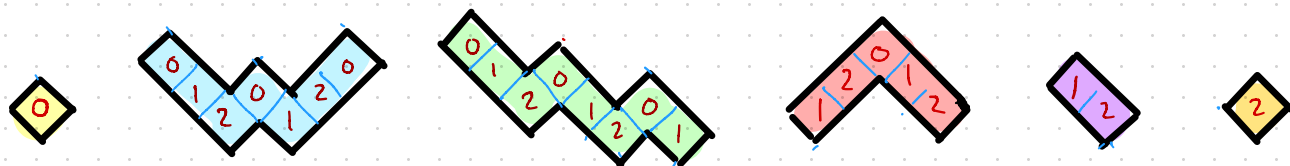


CUSPIDAL RIBBON TABLEAUX



ROB MUTH

WASHINGTON & JEFFERSON COLLEGE, PA USA

J/W D. ABBASIAN, L. DIFULVIO, G. PASTERNAK, I. SHOLTES & F. SINCLAIR

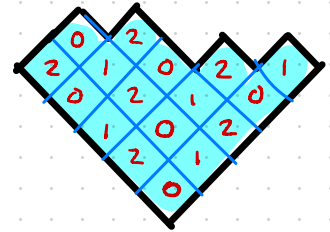
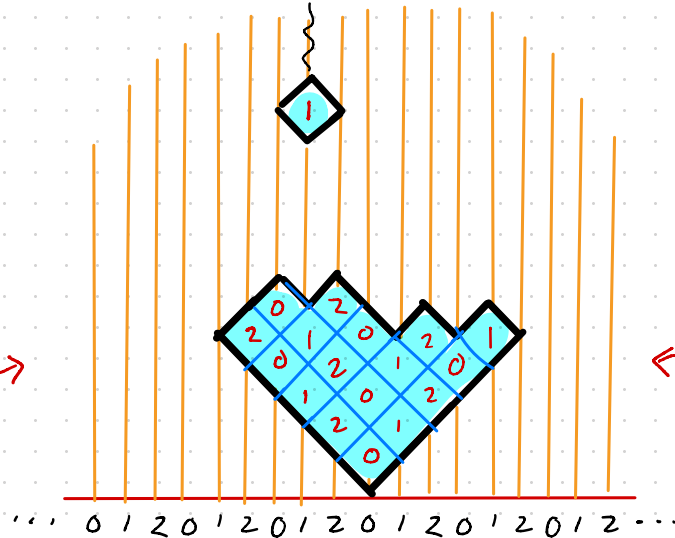
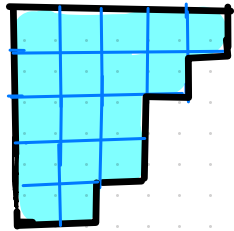
OIST REPRESENTATION THEORY SEMINAR

26/27 OCTOBER 2020

COMBINATORICS

SKREW SHAPES

- Fix $e > 1$.
- Young diagrams (Russian notation)

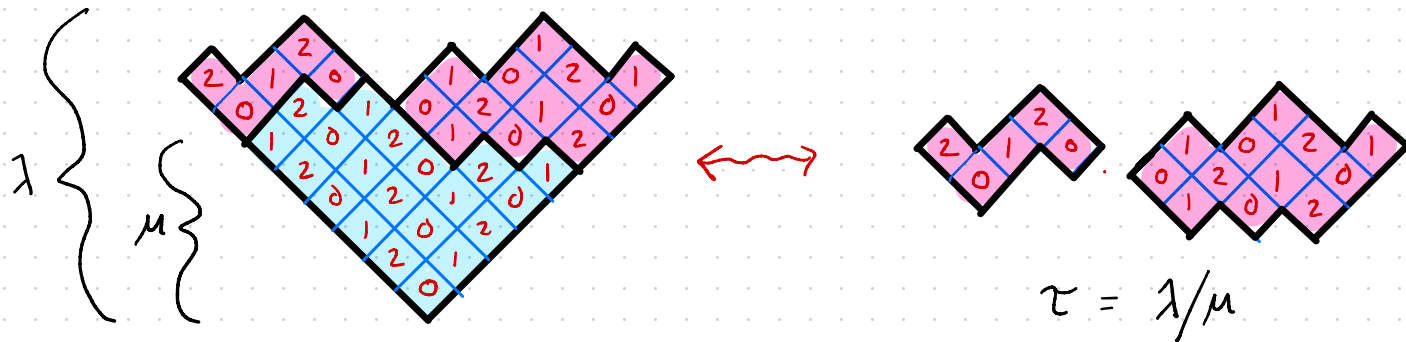


PARTITION
 $(5, 4, 3, 3, 2) \vdash 17$

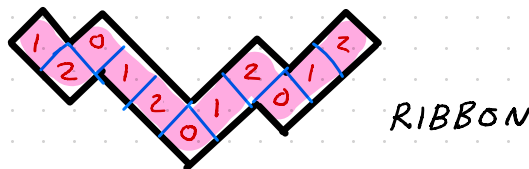
ABACUS w/ RUNNERS
 LABELED MOD e .

YOUNG DIAGRAM
 w/ RESIDUES SHOWN.

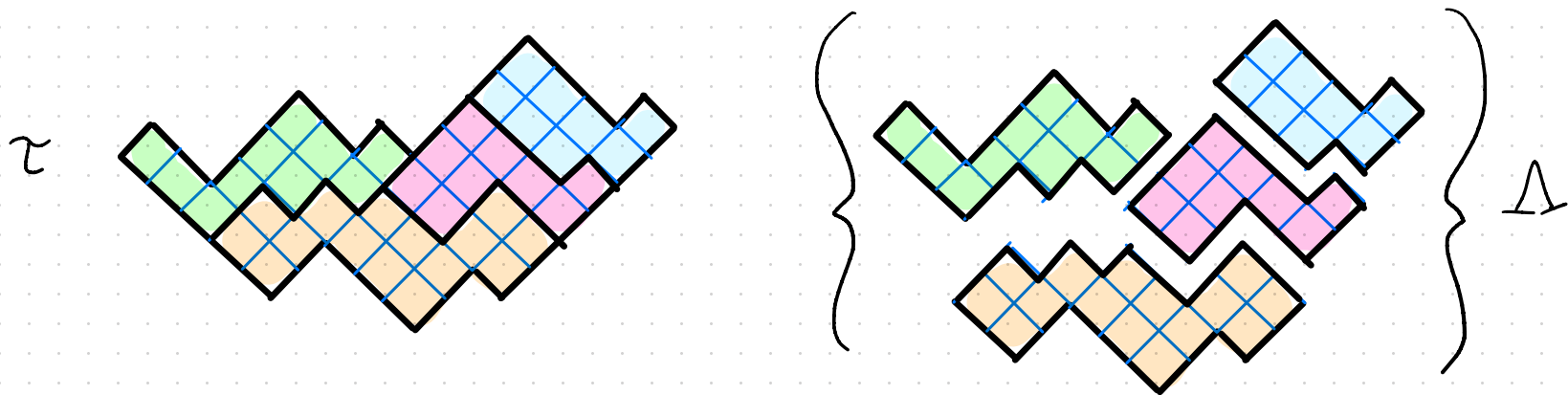
- A skew shape is a set difference of Young diagrams:



- A skew shape τ is connected if any two beads in τ may be connected by a path in τ .
- A ribbon is a connected skew shape with at most one bead on each runner:



- A **(skew) tiling** Δ of a skew shape τ is a collection of disjoint skew shapes whose union is τ :



- A **(skew) tableau** (Δ, t) of τ is a tiling Δ of τ with an ordering $t: \{1, 2, \dots, |\Delta|\} \rightarrow \Delta$ on tiles, such that tiles may be "dropped into position" in τ in this order.

* A Young tableau is a tableau where every tile is a single bead.

COMBINATORICS

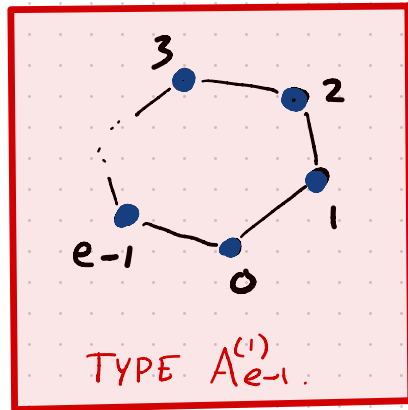
ROOT SYSTEMS

- Let $Q_+ = \mathbb{Z}\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{e-1}\}$ be the **root lattice**,
and Φ_+ be the set of **positive roots** of affine Lie type $A_{e-1}^{(1)}$.

- $\Phi_+ = \{ \alpha_k + \alpha_{k+1} + \dots + \alpha_{k+L} \mid k \in \mathbb{Z}_e, L \geq 0 \}$

- $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_{e-1}$ is the **null root**.

- $\Phi_+ = \Phi_+^{\text{re}} \sqcup \Phi_+^{\text{im}}$



Real roots
(height not divisible
by e)

Imaginary roots $\{m\delta \mid m \in \mathbb{N}\}$
(height divisible by e)

- $\Psi = \Phi_+^{\text{re}} \sqcup \{\delta\}$ is the set of **indivisible roots**.

- Choose a convex preorder $\succsim$ on Φ_+
 - total order on Ψ
 - if $\beta, \gamma \in \Phi_+$, $\beta \succsim \gamma$ and $\beta + \gamma \in \Phi_+$, then $\beta \succsim \beta + \gamma \succsim \gamma$.
 - all imaginary roots are equivalent under $\succsim$.

Ex ($e=2$) There are 2 possible convex preorders.

$$\alpha_1 \succ \delta + \alpha_1 \succ 2\delta + \alpha_1 \succ \dots \succ \delta \succ \dots \succ 2\delta + \alpha_0 \succ \delta + \alpha_0 \succ \alpha_0.$$

- For $\theta \in \mathbb{Q}_+$, a **Kostant partition** $\underline{\kappa}$ of θ is a tuple:

$$\underline{\kappa} = (\beta^{m_\beta})_{\beta \in \Psi} \quad \text{such that} \quad \theta = \sum_{\beta \in \Psi} m_\beta \beta.$$

- The convex preorder $\succsim$ induces a billexicographic partial order $\succeq$ on Kostant partitions of θ .

Ex ($e=2$) $\alpha_1 \succ \delta + \alpha_1 \succ 2\delta + \alpha_1 \succ \dots \succ \delta \succ \dots \succ 2\delta + \alpha_0 \succ \delta + \alpha_0 \succ \alpha_0$

$$\theta = 7\alpha_0 + 9\alpha_1 \quad \left\{ \begin{array}{l} \underline{\kappa}_1 = (\alpha_1^2 \mid 2\delta + \alpha_1 \mid \delta^2 \mid \delta + \alpha_0 \mid \alpha_0) \\ \underline{\kappa}_2 = (\alpha_1 \mid (\delta + \alpha_1)^2 \mid \delta \mid 3\delta + \alpha_0) \end{array} \right.$$

COMBINATORICS

ROOT SYSTEMS

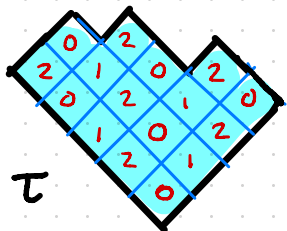
&

SKREW SHAPES

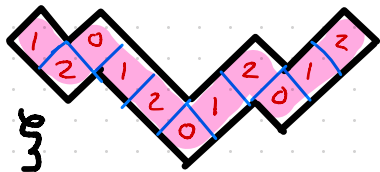
- For a skew shape τ , the **content** $\text{cont}(\tau) \in \mathbb{Q}_+$ is:

$$\sum_{\substack{\text{beads} \\ u \in \tau}} \alpha_{\text{residue}(u)} \in \mathbb{Q}_+$$

Ex ($e=3$)



$$\rightsquigarrow \text{cont}(\tau) = 6\alpha_0 + 4\alpha_1 + 6\alpha_2$$



$$\rightsquigarrow \text{cont}(\xi) = 3\alpha_0 + \alpha_1 + \alpha_2$$

* Ribbons have content in Φ_+ .

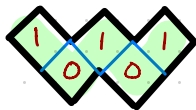
Def Let τ be a skew shape, and $\text{cont}(\tau) = \beta \in \Psi$

We say τ is **cuspidal** provided that for every two-part tableau (t_1, t_2) of τ , we have:

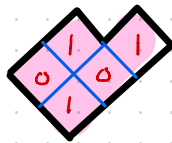
- o $\text{cont}(t_1)$ can be written as a sum of roots $< \beta$
- o $\text{cont}(t_2)$ can be written as a sum of roots $> \beta$.

Ex ($e=2$) $\alpha_1 > \delta + \alpha_1 > 2\delta + \alpha_1 > \dots > \delta > \dots > 2\delta + \alpha_0 > \delta + \alpha_0 > \alpha_0$.

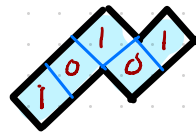
$$\beta = 2\delta + \alpha_1$$



CUSPIDAL



NON-CUSPIDAL

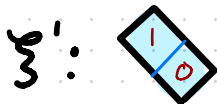
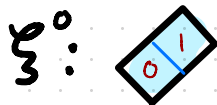


NON-CUSPIDAL

Theorem (ADMPSS)

- (a) Every cuspidal skew shape is a ribbon.
- (b) For $\beta \in \mathbb{Z}_+^{re}$, there exists a unique cuspidal ribbon ζ^β of content β .
- (c) There exist e distinct ribbons $\zeta^0, \zeta^1, \dots, \zeta^{e-1}$ of content δ .

Ex ($e=2$) $\alpha, \succ \delta + \alpha, \succ 2\delta + \alpha, \succ \dots \succ \delta \succ \dots \succ 2\delta + \alpha_0 \succ \delta + \alpha_0 \succ \alpha_0$



Ex ($e=3$). A less regular convex preorder.

Set a total order on $\mathbb{Q}^2$ via $(x,y) \geq (x',y') \Leftrightarrow \begin{cases} x > x', \text{ or} \\ x = x', y \geq y'. \end{cases}$

Define $h: \Phi_+ \rightarrow \mathbb{Q}^2$ by setting

$$h(\alpha_0) = (2,1) \quad h(\alpha_2) = (-1,0) \quad h(\alpha_2) = (-1,1),$$

and extend by linearity to Φ_+ . For $\beta, \gamma \in \Phi_+$, set

$$\beta \geq \gamma \Leftrightarrow \frac{h(\beta)}{ht(\beta)} \geq \frac{h(\gamma)}{ht(\gamma)}.$$

REAL:
(SAMPLING)

$\xi^\beta:$

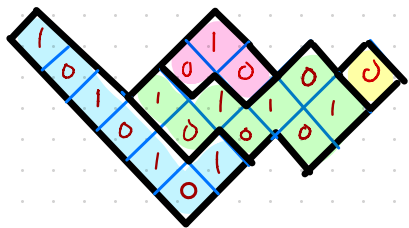
$$\alpha_0 > 2\alpha_0 > 2\alpha_0 + \alpha_1 > \alpha_0 + \alpha_1 + \alpha_2 > \alpha_1 + \alpha_2 > \alpha_2$$

IMAG:

$\xi^0:$ $\xi^1:$ $\xi^2:$

Def. A tableau (Λ, t) for a skew shape τ is **Kostant** if:

- $\text{cont}(\lambda) \in \Phi_+$ for all $\lambda \in \Lambda$
- $\text{cont}(t(1)) \geq \text{cont}(t(2)) \geq \dots \geq \text{cont}(t(\Lambda))$.
- A tiling Λ is **Kostant** if a Kostant Λ -tableau exists.
- A Kostant tiling Λ is naturally associated with a Kostant partition κ^Λ of $\text{cont}(\tau)$.



$$\longleftrightarrow \kappa^\Lambda = (3\delta + \alpha_1 \mid \delta^4 \mid \delta + \alpha_0 \mid \alpha_0)$$

- A tiling is **cuspidal** if all tiles are cuspidal.

Theorem (ADMPSS) Let τ be a skew shape.

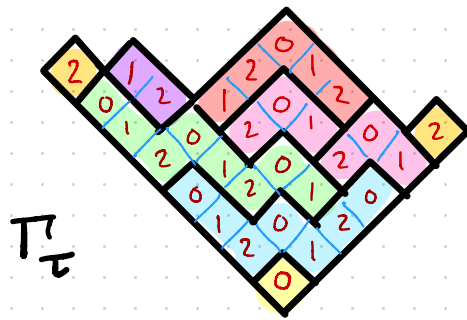
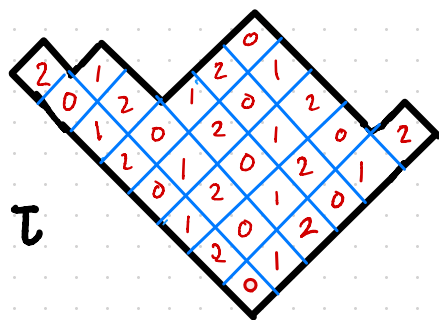
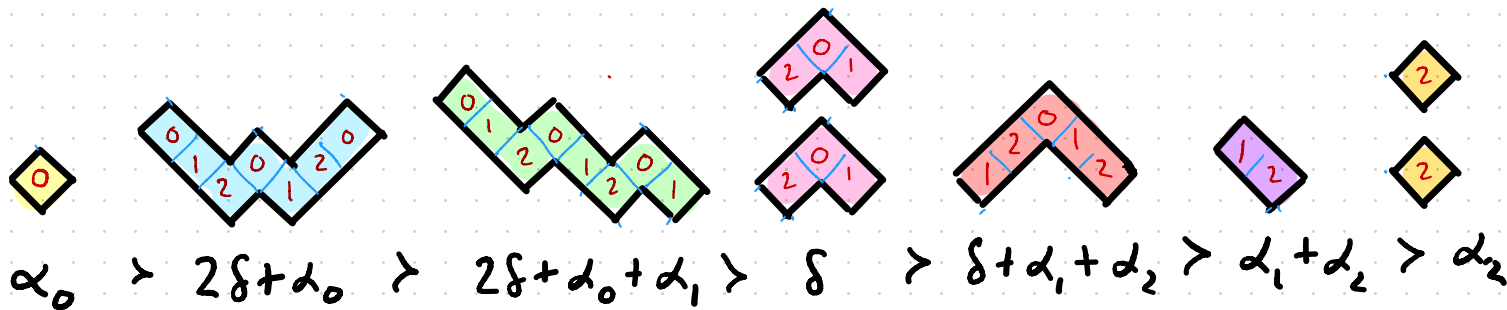
(a) τ has a unique cuspidal Kostant tiling T_τ .

(b) If Λ is any Kostant tiling for τ , then:

$$\kappa^{T_\tau} \supseteq \kappa^\Lambda.$$

(c) T_τ may be constructed by progressive removal of minimal ribbons.

Ex ($e=3$) CUSPIDAL SHAPES:



$$K^{\tau_\tau} = (\alpha_0 | 2\delta + \alpha_0 | 2\delta + \alpha_0 + \alpha_1 | \delta^2 | \delta + \alpha_1 + \alpha_2 | \alpha_1 + \alpha_2 | \alpha_2^2)$$

Def Let $m \in \mathbb{N}$, $\beta \in \Psi$.

Let τ be a skew shape with $\text{cont}(\tau) = m\beta$.

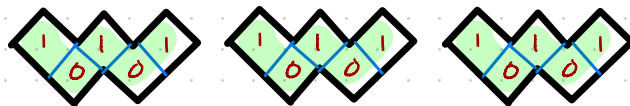
We say τ is **semicuspidal** provided that for every two-part tableau (t_1, t_2) of τ , we have:

- $\text{cont}(t_1)$ can be written as a sum of roots $\leq \beta$
- $\text{cont}(t_2)$ can be written as a sum of roots $\geq \beta$.

Theorem (ADMPSS) For $m \in \mathbb{N}$, $\beta \in \Phi^{\text{re}}$, there exists a unique semicuspidal skew shape $\mathfrak{S}^{m\beta}$ of content $m\beta$.

Ex ($e=2$)

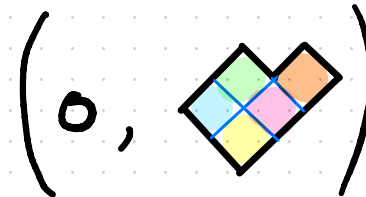
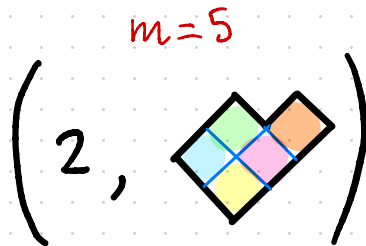
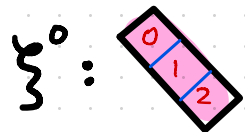
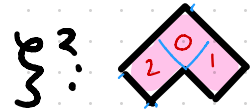
$$\mathfrak{S}^{3(2\delta + \alpha_1)} =$$



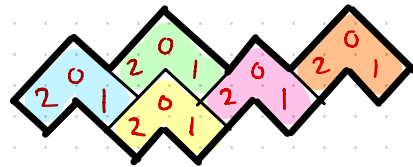
Theorem (ADMPSS) Let $m \in \mathbb{N}$, and let S_m^c be the set of connected skew shapes of size m . Then we have:

$$\mathbb{Z}_e \times S_m^c \xrightarrow[\text{i-dilation}]{\sim} \left\{ \begin{array}{c} \text{connected} \\ \text{semicuspidal skew shapes} \\ \text{of content } m\delta \end{array} \right\}$$

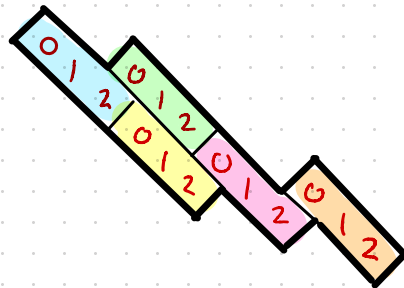
Ex ($e=3$)



$\xrightarrow[\text{2-dilation}]{\sim}$



$\xrightarrow[\text{0-dilation}]{\sim}$

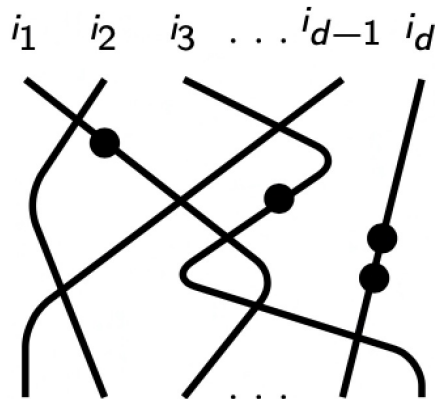


REPRESENTATION THEORY

QUIVER HECKE ALGEBRAS

- Keep $e, \mathbb{E}_+$ as before. Fix a field $\mathbb{k}$.
- Associated to every $\theta \in Q_+$ is a ($\mathbb{Z}$ -graded) associative **Quiver Hecke (KLR)** $\mathbb{k}$ -algebra R_θ .

SPANNED BY DIAGRAMS
OF THE FORM:



$i_1, i_2, \dots, i_d$ SIMPLE ROOTS

GENERATORS

$$1_i = \begin{array}{|c|} \hline i_1 \ i_2 \ \dots \ i_n \\ \hline \end{array}$$

$$y_r 1_i = \begin{array}{|c|} \hline i_1 \ \dots \ i_r \ \dots \ i_n \\ \hline \bullet \\ \hline \end{array}$$

$$\psi_r 1_i = \begin{array}{|c|} \hline i_1 \ \dots \ i_r \ i_{r+1} \ \dots \ i_n \\ \hline \times \\ \hline \end{array}$$

RELATIONS

$$\begin{array}{l} \begin{array}{|c|} \hline i \ j \\ \hline \end{array} = \begin{cases} \varepsilon_{ij} \left(\begin{array}{|c|} \hline i \ j \\ \hline \bullet \end{array} - \begin{array}{|c|} \hline i \ j \\ \hline \bullet \end{array} \right) & c_{i,j} = -1; \\ 0 & i = j; \\ \begin{array}{|c|} \hline i \ j \\ \hline \end{array} & \text{otherwise,} \end{cases}$$

$$\begin{array}{|c|} \hline i \ j \ k \\ \hline \end{array} - \begin{array}{|c|} \hline i \ j \ k \\ \hline \end{array} = \begin{cases} \varepsilon_{ij} \begin{array}{|c|} \hline i \ j \ i \\ \hline \end{array} & i = k, c_{i,j} = -1; \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{array}{|c|} \hline i \ j \\ \hline \end{array} - \begin{array}{|c|} \hline i \ j \\ \hline \end{array} = \delta_{i,j} \begin{array}{|c|} \hline i \ i \\ \hline \end{array} = \begin{array}{|c|} \hline i \ j \\ \hline \end{array} - \begin{array}{|c|} \hline i \ j \\ \hline \end{array}.$$

Pertinent facts about R_θ -mod

- For $\theta_1, \dots, \theta_k \in Q_+$, there is an inclusion:

$$R_{\theta_1} \otimes \dots \otimes R_{\theta_k} \hookrightarrow R_{\theta_1 + \dots + \theta_k}$$

- We have exact induction and restriction functors:

$$\text{Ind}_{\theta_1, \theta_2}^{\theta_1 + \theta_2} : R_{\theta_1} \otimes R_{\theta_2}\text{-mod} \longrightarrow R_{\theta_1 + \theta_2}\text{-mod}.$$

$$\text{Res}_{\theta_1, \theta_2}^{\theta_1 + \theta_2} : R_{\theta_1 + \theta_2}\text{-mod} \longrightarrow R_{\theta_1} \otimes R_{\theta_2}\text{-mod}.$$

- Write $M \circ N := \text{Ind}_{\theta_1, \theta_2}^{\theta_1 + \theta_2}(M \boxtimes N)$.

The following definition motivated our definition of cuspidality for skew shapes:

Def • Let $\beta \in \Psi$, $m \in \mathbb{N}$. We say $M \in R_{m\beta}\text{-mod}$ is:
semicuspidal provided:

$$\text{Res}_{\mu, \nu}^{m\beta} M \neq 0 \Rightarrow \begin{cases} \mu \text{ is a sum of roots } \leq \beta \\ \nu \text{ is a sum of roots } \geq \beta \end{cases}$$

- We say M is **cuspidal** if $m=1$ and the comparisons may be made strict.

Theorem (Kleshchev) Let $m \in \mathbb{N}$, and $\beta \in \Phi_+^{re}$

- There exists a unique simple semicuspidal module $L(\beta^m) \in R_{m\beta}\text{-mod}$.
- The simple semicuspidal $R_{m\beta}$ -modules may be labeled:
 $\{ L(\underline{\lambda}) \mid \underline{\lambda} \text{ an } (e-1) \text{ multipartition of } m \}$

- Quiver Hecke algebras, introduced by Khovanov-Lauda and Rouquier (2008) categorify quantum groups:

$$\bigoplus_{\theta \in Q_+} [\text{Proj}(R_\theta)] \xrightarrow{\sim} \mathcal{U}_q^+(\hat{\mathfrak{sl}}_e)$$

$$\bigoplus_{\theta \in Q_+} [\text{Rep}_{\text{fd}}(R_\theta)] \xrightarrow{\sim} \mathcal{U}_q^+(\hat{\mathfrak{sl}}_e)^*$$

Bases in $U_q^+(\mathfrak{g})$ & $U_q^+(\mathfrak{g})^*$

canonical basis

PBW basis

$\cup$
root vectors

dual canonical basis

dual PBW basis

$\cup$
dual root vectors

Families in $R_\beta\text{-mod.}$

projective indecomposables

standard modules

$\cup$
semicuspidal standards

simples

proper standard modules

$\cup$
semicuspidal simples

- A **root partition** $(\kappa, \underline{\lambda})$ of $\theta \in \mathbb{Q}_+$ is the data:
 - a Kostant partition $\kappa = (\beta^{m_\beta})_{\beta \in \Psi}$ of θ
 - an $(e-1)$ -multipartition $\underline{\lambda}$ of m_δ .
- The **proper standard module** $\Delta(\kappa, \underline{\lambda})$ is the induction product of the associated semicuspidals:

$$\Delta(\kappa, \underline{\lambda}) = L(\beta_1^{m_{\beta_1}}) \circ \cdots \circ L(\underline{\lambda}) \circ \cdots \circ L(\beta_k^{m_{\beta_k}})$$

Theorem (Kleshchev) $\Delta(\kappa, \underline{\lambda})$ has simple head $L(\kappa, \underline{\lambda})$.

$\{ L(\kappa, \underline{\lambda}) \mid (\kappa, \underline{\lambda}) \text{ a root partition of } \theta \}$
 is a complete set of simple R_θ -modules.

REPRESENTATION THEORY

SPECHT MODULES

- To any skew shape τ of content θ , there is an associated (row) **Specht module** $S^\tau \in R_\theta\text{-mod}$.
 – character/dimension can be read off from residue sequences of Young τ -tableaux.
- Specht modules arise as cell modules in the cellular structure of the cyclotomic quotient R_θ^Λ .
 (Hu-Mathas)

Ex If $\text{char } k = e$ is prime then

$$\bigoplus_{ht(\theta)=d} R_\theta^\Lambda \cong k\mathbb{G}_d.$$

Prop (ADMPSS) Let $\beta \in \Psi$, $m \in \mathbb{N}$.

Let τ be a skew shape of content $m\beta$.

Then S^τ is a (semi)cuspidal $R_{m\beta}$ -module
if and only if τ is a (semi)cuspidal skew shape.

Prop (ADMPSS) Let $\beta \in \Phi_+^{re}$, $m \in \mathbb{N}$. Then $L(\beta^m) \cong S^{\tilde{3}^m \beta}$.

* This gives a presentation/character of real semicuspidals.

Conj The simple semicuspidal $R_{m\beta}$ -modules arise
as the head of certain (explicit)
semicuspidal Specht modules S^τ .

TILINGS & SIMPLE FACTORS OF SPECHT MODULES

Prop (ADMPSS) Let τ be a skew shape, and let T_τ be the unique cuspidal Kostant tiling of τ .

(a) S^τ has a simple factor of the form

$$L(\kappa^{T_\tau}, \underline{1}).$$

(b) If $L(\kappa, \underline{\mu})$ is any simple factor of S^τ , then:

$$\kappa^{T_\tau} \supseteq \kappa.$$

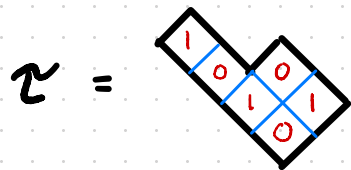
2 sets of labels for simple R_θ -modules:

D^τ
simple head of S
where τ is a
certain multipartition
of content θ

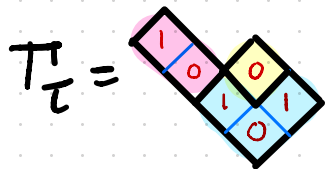
$L(\kappa, \underline{1})$
simple head of $\Delta(\kappa, \underline{1})$
where $(\kappa, \underline{1})$ is a
root partition of θ ,

$\tau \longleftrightarrow (\kappa, \underline{1})$
????

Ex ($e=2$) $\theta = 3\delta$.



$(2^2, 1^2)$



$$K^{\tau_\tau} = (\delta + \alpha_1 | \delta | \alpha_0)$$

$$S^\tau \cong \left\{ \begin{array}{l} \frac{D^{(2^2, 1^2)}}{D^{(2, 1^4)}} \longleftrightarrow L((\delta + \alpha_1 | \delta + \alpha_0), \emptyset) \\ \frac{D^{(2, 1^4)}}{D^{(1^6)}} \longleftrightarrow L((\delta + \alpha_1 | \delta | \alpha_0), (1)) \\ \frac{D^{(1^6)}}{D^{(1^6)}} \longleftrightarrow L(3\delta, (3)). \end{array} \right.$$

ONGOING WORK / FUTURE QUESTIONS

- RoCK blocks (and Scopes equivalent blocks), and semicuspidal imaginary Specht modules.
- Core tilings (and (e, k) -cores)
- Cuspidal ribbon tableaux in other types.

Thank You!