On stable modular plethysms of the natural module of $SL_2(\mathbb{F}_p)$ in characteristic p

Pavel Turek

Royal Holloway, University of London

January 24, 2023

Based on arXiv:2210.08943

Outline

1. Schur functors and modular plethysms
2. Representation theory of $SL_2(\mathbb{F}_p)$ in characteristic p
3. Computation of modular plethysms
4. Endotrivial modules and Schur functors

Outline

1. Schur functors and modular plethysms
2. Representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$ in characteristic p
3. Computation of modular plethysms
4. Endotrivial modules and Schur functors

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition.

Young diagrams and tableaux

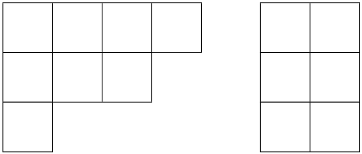
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda),\ j \leq \lambda_i\right\}$.

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda),\ j \leq \lambda_i\right\}$. Given a set S we define a λ-tableau (with entries in S) to be a map $t : Y(\lambda) \to S$.

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda),\ j \leq \lambda_i\}$. Given a set S we define a λ-tableau (with entries in S) to be a map $t : Y(\lambda) \to S$.



These are the Young diagrams of $(4, 3, 1)$ and (2^3).

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda),\ j \leq \lambda_i\right\}$. Given a set S we define a λ-tableau (with entries in S) to be a map $t : Y(\lambda) \to S$.

1	2	3	3
2	2	4	
4			

1	2
3	4
6	6

These are the Young diagrams of $(4,3,1)$ and (2^3). And now tableaux (with entries in $\mathbb{N}$).

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\left\{(i,j) \in \mathbb{N}^2 : i \le \ell(\lambda),\ j \le \lambda_i\right\}$. Given a set S we define a λ-tableau (with entries in S) to be a map $t : Y(\lambda) \to S$.

1	2	3	3
2	2	4	
4			

1	2
3	4
6	6

These are the Young diagrams of $(4,3,1)$ and (2^3). And now tableaux (with entries in $\mathbb{N}$).

Suppose that S is totally ordered. We call a tableau t standard if its entries are weakly increasing along its rows and strictly increasing along its columns.

Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. We denote by $Y(\lambda)$ its Young diagram $\left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda),\ j \leq \lambda_i\right\}$. Given a set S we define a λ-tableau (with entries in S) to be a map $t : Y(\lambda) \to S$.

1	2	3	3
2	2	4	
4			

1	2
3	4
6	6

These are the Young diagrams of $(4, 3, 1)$ and (2^3). And now tableaux (with entries in $\mathbb{N}$).

Suppose that S is totally ordered. We call a tableau t standard if its entries are weakly increasing along its rows and strictly increasing along its columns. The first tableau is not standard, while the second is.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\operatorname{Sym}^{\lambda} V := \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^{\lambda} V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^{\lambda} V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w & \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

For a tableau t define $e(t) = \sum_{\tau \in C(\lambda)} \mathrm{sgn}(\tau)\tau \cdot t$, where $C(\lambda)$ is the column stabiliser in the group of box permutation.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^{\lambda} V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

For a tableau t define $e(t) = \sum_{\tau \in C(\lambda)} \mathrm{sgn}(\tau)\tau \cdot t$, where $C(\lambda)$ is the column stabiliser in the group of box permutation.

Example:

$$e\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline d & b & c \\ \hline a & e \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline a & e & c \\ \hline d & b \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline d & e & c \\ \hline a & b \\ \cline{1-2}\end{array}$$

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^{\lambda} V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

For a tableau t define $e(t) = \sum_{\tau \in C(\lambda)} \mathrm{sgn}(\tau)\tau \cdot t$, where $C(\lambda)$ is the column stabiliser in the group of box permutation.

Example:

$$e\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline d & b & c \\ \hline a & e \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline a & e & c \\ \hline d & b \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline d & e & c \\ \hline a & b \\ \cline{1-2}\end{array}$$

The subspace of $\mathrm{Sym}^{\lambda} V$ spanned by $e(t)$ is a $K\mathrm{GL}(V)$-submodule $\nabla^{\lambda} V$. We refer to ∇^{λ} as the Schur functor labelled by λ.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^\lambda V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w & \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

For a tableau t define $e(t) = \sum_{\tau \in C(\lambda)} \mathrm{sgn}(\tau)\tau \cdot t$, where $C(\lambda)$ is the column stabiliser in the group of box permutation.

Example:

$$e\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline d & b & c \\ \hline a & e & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline a & e & c \\ \hline d & b & \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline d & e & c \\ \hline a & b & \\ \cline{1-2}\end{array}$$

The subspace of $\mathrm{Sym}^\lambda V$ spanned by $e(t)$ is a $K\mathrm{GL}(V)$-submodule $\nabla^\lambda V$. We refer to ∇^λ as the Schur functor labelled by λ. We have $\nabla^{(n)} = \mathrm{Sym}^n$ and $\nabla^{(1^n)} = \bigwedge^n$.

Schur functors

Let $S = V$ be a vector space over K. We identify λ-tableaux with entries in V with elements of $\mathrm{Sym}^\lambda V := \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{\ell(\lambda)}} V$ by 'multiplying along rows'. The tableau $\begin{array}{|c|c|}\hline u & v \\ \hline w & \\ \cline{1-1}\end{array}$ is identified with $uv \otimes w \in \mathrm{Sym}^{(2,1)} V$.

For a tableau t define $e(t) = \sum_{\tau \in C(\lambda)} \mathrm{sgn}(\tau)\tau \cdot t$, where $C(\lambda)$ is the column stabiliser in the group of box permutation.

Example:

$$e\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline d & b & c \\ \hline a & e & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline a & e & c \\ \hline d & b & \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline d & e & c \\ \hline a & b & \\ \cline{1-2}\end{array}$$

The subspace of $\mathrm{Sym}^\lambda V$ spanned by $e(t)$ is a $K\mathrm{GL}(V)$-submodule $\nabla^\lambda V$. We refer to ∇^λ as the Schur functor labelled by λ. We have $\nabla^{(n)} = \mathrm{Sym}^n$ and $\nabla^{(1^n)} = \bigwedge^n$. If K has characteristic zero, $\nabla^\lambda V$ give all the indecomposable polynomial $K\mathrm{GL}(V)$-modules.

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \dots$.

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \ldots$. For a partition λ, the Schur function s_λ is given by

$$s_\lambda = \sum_{T \in \mathrm{Std}(\lambda)} x^T.$$

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \ldots$. For a partition λ, the Schur function s_λ is given by

$$s_\lambda = \sum_{T \in \mathrm{Std}(\lambda)} x^T.$$

Example:

$$s_{(2,1)} = x^{\ytableaushort{11,2}} + x^{\ytableaushort{11,3}} + x^{\ytableaushort{12,2}} + x^{\ytableaushort{12,3}} + x^{\ytableaushort{13,2}} + x^{\ytableaushort{13,3}} + x^{\ytableaushort{22,3}} + x^{\ytableaushort{23,3}} + \ldots = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots.$$

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \dots$. For a partition λ, the Schur function s_λ is given by

$$s_\lambda = \sum_{T \in \mathrm{Std}(\lambda)} x^T.$$

Example:

$$s_{(2,1)} = x^{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3&\end{smallmatrix}} + \dots = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \dots.$$

By omitting the three dots at the end we obtain $s_{(2,1)}(x_1, x_2, x_3)$.

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \ldots$. For a partition λ, the Schur function s_λ is given by

$$s_\lambda = \sum_{T \in \mathrm{Std}(\lambda)} x^T.$$

Example:

$$s_{(2,1)} = x^{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3&\end{smallmatrix}} + \ldots = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \ldots.$$

By omitting the three dots at the end we obtain $s_{(2,1)}(x_1, x_2, x_3)$.

We move between Schur functors and Schur functions via formal characters. The formal character of $\nabla^\lambda V$ is $s_\lambda(x_1, x_2, \ldots, x_d)$ where $d = \dim V$.

Schur functions

Let Λ be a ring of symmetric functions in variables $x_1, x_2, \ldots$. For a partition λ, the Schur function s_λ is given by

$$s_\lambda = \sum_{T \in \mathrm{Std}(\lambda)} x^T.$$

Example:

$$s_{(2,1)} = x^{\ytableaushort{11,2}} + x^{\ytableaushort{11,3}} + x^{\ytableaushort{12,2}} + x^{\ytableaushort{12,3}} + x^{\ytableaushort{13,2}} + x^{\ytableaushort{13,3}} + x^{\ytableaushort{22,3}} + x^{\ytableaushort{23,3}} + \ldots = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \ldots.$$

By omitting the three dots at the end we obtain $s_{(2,1)}(x_1, x_2, x_3)$.

We move between Schur functors and Schur functions via formal characters. The formal character of $\nabla^\lambda V$ is $s_\lambda(x_1, x_2, \ldots, x_d)$ where $d = \dim V$. This is particularly useful in characteristic zero or p with p larger than the underlying polynomial degree.

Plethysms and $\mathrm{SL}_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors.

Plethysms and $SL_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Plethysms and $SL_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x,y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Plethysms and $SL_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x,y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Stanley has identified the problem of decomposing plethysms as one of the major open problems in algebraic combinatorics.

Plethysms and $SL_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x, y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Stanley has identified the problem of decomposing plethysms as one of the major open problems in algebraic combinatorics.

For the modular plethysms $\nabla^\nu \nabla^\mu$ of the natural $SL_2(\mathbb{F}_p)$-module over characteristic p we can assume that $\mu = (l)$.

Plethysms and $SL_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x, y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Stanley has identified the problem of decomposing plethysms as one of the major open problems in algebraic combinatorics.

For the modular plethysms $\nabla^\nu \nabla^\mu$ of the natural $SL_2(\mathbb{F}_p)$-module over characteristic p we can assume that $\mu = (l)$. Throughout we moreover assume that $l \leq p - 2$ and $|\nu| < p$ (we say ν is p-small).

Plethysms and $\mathrm{SL}_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x,y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Stanley has identified the problem of decomposing plethysms as one of the major open problems in algebraic combinatorics.

For the modular plethysms $\nabla^\nu \nabla^\mu$ of the natural $\mathrm{SL}_2(\mathbb{F}_p)$-module over characteristic p we can assume that $\mu = (l)$. Throughout we moreover assume that $l \leq p-2$ and $|\nu| < p$ (we say ν is p-small).

In the literature:

- computational formulas for modular plethysms when ∇^ν is a symmetric or an exterior power established by Kouwenhoven in 1990,

Plethysms and $\mathrm{SL}_2(\mathbb{F}_p)$

A modular plethysm is a composition of two Schur functors. Taking formal characters we obtain plethysms of Schur functions.

Example:

$$s_{(2,1)} \circ s_{(2)}(x,y) = s_{(2,1)}(x^2, xy, y^2) = x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5$$

Stanley has identified the problem of decomposing plethysms as one of the major open problems in algebraic combinatorics.

For the modular plethysms $\nabla^\nu \nabla^\mu$ of the natural $\mathrm{SL}_2(\mathbb{F}_p)$-module over characteristic p we can assume that $\mu = (l)$. Throughout we moreover assume that $l \leq p-2$ and $|\nu| < p$ (we say ν is p-small).

In the literature:

- computational formulas for modular plethysms when ∇^ν is a symmetric or an exterior power established by Kouwenhoven in 1990,
- various isomorphisms of modular plethysms established by McDowell and Wildon in 2022.

Outline

1. Schur functors and modular plethysms
2. Representation theory of $SL_2(\mathbb{F}_p)$ in characteristic p
3. Computation of modular plethysms
4. Endotrivial modules and Schur functors

Dictionary

- p is an odd prime,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,
- G denotes the group $\mathrm{SL}_2(\mathbb{F}_p)$,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,
- G denotes the group $\mathrm{SL}_2(\mathbb{F}_p)$,
- E is the natural two dimensional kG-module,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,
- G denotes the group $\mathrm{SL}_2(\mathbb{F}_p)$,
- E is the natural two dimensional kG-module,
- H denotes an arbitrary finite group of order divisible by p,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,
- G denotes the group $\mathrm{SL}_2(\mathbb{F}_p)$,
- E is the natural two dimensional kG-module,
- H denotes an arbitrary finite group of order divisible by p,
- $\cong$ denotes an isomorphism of kH-modules,

Dictionary

- p is an odd prime,
- k is a field of characteristic p,
- ζ_p is a primitive (complex) pth root of unity,
- G denotes the group $\mathrm{SL}_2(\mathbb{F}_p)$,
- E is the natural two dimensional kG-module,
- H denotes an arbitrary finite group of order divisible by p,
- $\cong$ denotes an isomorphism of kH-modules,
- $\overset{p}{\cong}$ denotes an isomorphism in the stable module category of kH ('isomorphism modulo projectives'). That is, for two kH-modules V, W we write $V \overset{p}{\cong} W$ if there are projective kH-modules P, Q such that $V \oplus P \cong W \oplus Q$.

Irreducible modules

The irreducible kG-modules are given by $\mathrm{Sym}^0 E, \mathrm{Sym}^1 E, \ldots, \mathrm{Sym}^{p-1} E$.

Irreducible modules

The irreducible kG-modules are given by $\mathrm{Sym}^0 E, \mathrm{Sym}^1 E, \dots, \mathrm{Sym}^{p-1} E$.
They are all non-projective apart from $\mathrm{Sym}^{p-1} E$.

Irreducible modules

The irreducible kG-modules are given by $\operatorname{Sym}^0 E, \operatorname{Sym}^1 E, \ldots, \operatorname{Sym}^{p-1} E$. They are all non-projective apart from $\operatorname{Sym}^{p-1} E$.

Theorem (Stable Clebsch–Gordan rule)

Let $0 \le i \le j \le p-2$. Writing S for the tensor product $\operatorname{Sym}^i E \otimes \operatorname{Sym}^j E$ we have the decomposition

$$S \overset{p}{\cong} \begin{cases} \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{i+j} E & \text{if } i+j < p-2, \\ \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{2p-4-i-j} E & \text{if } i+j \ge p-2. \end{cases}$$

Irreducible modules

The irreducible kG-modules are given by $\operatorname{Sym}^0 E, \operatorname{Sym}^1 E, \dots, \operatorname{Sym}^{p-1} E$. They are all non-projective apart from $\operatorname{Sym}^{p-1} E$.

Theorem (Stable Clebsch–Gordan rule)

Let $0 \leq i \leq j \leq p-2$. Writing S for the tensor product $\operatorname{Sym}^i E \otimes \operatorname{Sym}^j E$ we have the decomposition

$$S \overset{p}{\cong} \begin{cases} \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{i+j} E & \text{if } i+j < p-2, \\ \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{2p-4-i-j} E & \text{if } i+j \geq p-2. \end{cases}$$

Observe that all the non-projective indecomposable summands of a tensor product of two irreducible kG-modules are irreducible.

Irreducible modules

The irreducible kG-modules are given by $\mathrm{Sym}^0 E, \mathrm{Sym}^1 E, \ldots, \mathrm{Sym}^{p-1} E$. They are all non-projective apart from $\mathrm{Sym}^{p-1} E$.

Theorem (Stable Clebsch–Gordan rule)

Let $0 \leq i \leq j \leq p-2$. Writing S for the tensor product $\mathrm{Sym}^i E \otimes \mathrm{Sym}^j E$ we have the decomposition

$$S \overset{p}{\cong} \begin{cases} \mathrm{Sym}^{j-i} E \oplus \mathrm{Sym}^{j-i+2} E \oplus \cdots \oplus \mathrm{Sym}^{i+j} E & \text{if } i+j < p-2, \\ \mathrm{Sym}^{j-i} E \oplus \mathrm{Sym}^{j-i+2} E \oplus \cdots \oplus \mathrm{Sym}^{2p-4-i-j} E & \text{if } i+j \geq p-2. \end{cases}$$

Observe that all the non-projective indecomposable summands of a tensor product of two irreducible kG-modules are irreducible.

Example: $\mathrm{Sym}^3 E \otimes \mathrm{Sym}^3 E \overset{p}{\cong} \mathrm{Sym}^0 E \oplus \mathrm{Sym}^2 E \oplus \mathrm{Sym}^4 E \oplus \mathrm{Sym}^6 E$ for $p \geq 11$.

Irreducible modules

The irreducible kG-modules are given by $\operatorname{Sym}^0 E, \operatorname{Sym}^1 E, \dots, \operatorname{Sym}^{p-1} E$. They are all non-projective apart from $\operatorname{Sym}^{p-1} E$.

Theorem (Stable Clebsch–Gordan rule)

Let $0 \leq i \leq j \leq p-2$. Writing S for the tensor product $\operatorname{Sym}^i E \otimes \operatorname{Sym}^j E$ we have the decomposition

$$S \overset{p}{\cong} \begin{cases} \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{i+j} E & \text{if } i+j < p-2, \\ \operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{2p-4-i-j} E & \text{if } i+j \geq p-2. \end{cases}$$

Observe that all the non-projective indecomposable summands of a tensor product of two irreducible kG-modules are irreducible.

Example: $\operatorname{Sym}^3 E \otimes \operatorname{Sym}^3 E \overset{p}{\cong} \operatorname{Sym}^0 E \oplus \operatorname{Sym}^2 E \oplus \operatorname{Sym}^4 E \oplus \operatorname{Sym}^6 E$ for $p \geq 11$. If instead $p = 5$, then $\operatorname{Sym}^3 E \otimes \operatorname{Sym}^3 E \overset{p}{\cong} \operatorname{Sym}^0 E$.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V. We refer to Ω as the Heller operator.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V. We refer to Ω as the Heller operator.

Lemma

Let k denote the trivial kH-module. For any kH-module V we have $\Omega k \otimes V \overset{p}{\cong} \Omega V$.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V. We refer to Ω as the Heller operator.

Lemma

Let k denote the trivial kH-module. For any kH-module V we have $\Omega k \otimes V \overset{p}{\cong} \Omega V$.

It can be shown that the non-projective indecomposable kG-modules are labelled by pairs (l, m) with $0 \leq l \leq p-2$ and $m \in \mathbb{Z}/(p-1)\mathbb{Z}$. The pair (l, m) corresponds to the module $\Omega^m\left(\mathrm{Sym}^l E\right)$.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V. We refer to Ω as the Heller operator.

Lemma

Let k denote the trivial kH-module. For any kH-module V we have $\Omega k \otimes V \overset{p}{\cong} \Omega V$.

It can be shown that the non-projective indecomposable kG-modules are labelled by pairs (l, m) with $0 \leq l \leq p-2$ and $m \in \mathbb{Z}/(p-1)\mathbb{Z}$. The pair (l, m) corresponds to the module $\Omega^m \left(\mathrm{Sym}^l E\right)$.
We can tensor two such modules by adding powers of Ω and using the Clebsch–Gordan rule.

Indecomposable modules

Recall that for any kH-module V we define a module ΩV to be the kernel of the surjection $PV \twoheadrightarrow V$, where PV is the projective cover of V. We refer to Ω as the Heller operator.

Lemma

Let k denote the trivial kH-module. For any kH-module V we have $\Omega k \otimes V \overset{p}{\cong} \Omega V$.

It can be shown that the non-projective indecomposable kG-modules are labelled by pairs (l, m) with $0 \leq l \leq p-2$ and $m \in \mathbb{Z}/(p-1)\mathbb{Z}$. The pair (l, m) corresponds to the module $\Omega^m\left(\operatorname{Sym}^l E\right)$.

We can tensor two such modules by adding powers of Ω and using the Clebsch–Gordan rule.

Example: Let $p = 5$. We compute $\Omega\left(\operatorname{Sym}^3 E\right) \otimes \Omega^2\left(\operatorname{Sym}^3 E\right) \overset{p}{\cong} \Omega^3\left(\operatorname{Sym}^0 E\right)$.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

The isomorphism classes of the (virtual) projective modules form an ideal I. Let $\overline{R(G)} = R(G)/I$.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

The isomorphism classes of the (virtual) projective modules form an ideal I. Let $\overline{R(G)} = R(G)/I$. We write $\overline{V}$ for the element of $\overline{R(G)}$ corresponding to a kG-module V.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

The isomorphism classes of the (virtual) projective modules form an ideal I. Let $\overline{R(G)} = R(G)/I$. We write $\overline{V}$ for the element of $\overline{R(G)}$ corresponding to a kG-module V.

Theorem (T, 2022)

The map $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ given by $\zeta_p^2 + \zeta_p^{-2} \mapsto \overline{\mathrm{Sym}^2 E} - \overline{k}$, $X \mapsto \overline{\Omega k}$ and $Y \mapsto \overline{\mathrm{Sym}^{p-2} E}$ is an isomorphism.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

The isomorphism classes of the (virtual) projective modules form an ideal I. Let $\overline{R(G)} = R(G)/I$. We write $\overline{V}$ for the element of $\overline{R(G)}$ corresponding to a kG-module V.

Theorem (T, 2022)

The map $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ given by $\zeta_p^2 + \zeta_p^{-2} \mapsto \overline{\mathrm{Sym}^2 E} - \overline{k}$, $X \mapsto \overline{\Omega k}$ and $Y \mapsto \overline{\mathrm{Sym}^{p-2} E}$ is an isomorphism.

The subring R_I of $\overline{R(G)}$ consisting of the (virtual) semisimple modules corresponds under Ψ to $\mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$.

Representation ring

Write $R(G)$ for the representation ring of G, an abelian group generated freely by the isomorphism classes of the indecomposable kG-modules equipped with multiplication given by tensoring.

The isomorphism classes of the (virtual) projective modules form an ideal I. Let $\overline{R(G)} = R(G)/I$. We write $\overline{V}$ for the element of $\overline{R(G)}$ corresponding to a kG-module V.

Theorem (T, 2022)

The map $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ given by $\zeta_p^2 + \zeta_p^{-2} \mapsto \overline{\mathrm{Sym}^2 E} - \overline{k}$, $X \mapsto \overline{\Omega k}$ and $Y \mapsto \overline{\mathrm{Sym}^{p-2} E}$ is an isomorphism.

The subring R_I of $\overline{R(G)}$ consisting of the (virtual) semisimple modules corresponds under Ψ to $\mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. A further subring generated by even symmetric powers of E (up to $p - 3$) corresponds to $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.

Diagrammatic representation ring

Let us construct two $(p-1) \times (p-1)/2$ tables (for $p = 5$).

Diagrammatic representation ring

$h \backslash c$	0	1
3		
2		
1		
0		

$h \backslash c$	0	1
3		
2		
1		
0		

Let us construct two $(p-1) \times (p-1)/2$ tables (for $p = 5$). Label rows by $h \in \mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0 \leq c \leq (p-3)/2$.

Diagrammatic representation ring

$h\backslash c$	0	1
3	$\Omega^3 k$	$\Omega^3\left(\mathrm{Sym}^2 E\right)$
2	$\Omega^2 k$	$\Omega^2\left(\mathrm{Sym}^2 E\right)$
1	Ωk	$\Omega\left(\mathrm{Sym}^2 E\right)$
0	k	$\mathrm{Sym}^2 E$

0	1

Let us construct two $(p-1)\times(p-1)/2$ tables (for $p=5$). Label rows by $h\in\mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0\le c\le(p-3)/2$. We fill the first table with $\Omega^h\left(\mathrm{Sym}^{2c} E\right)$

Diagrammatic representation ring

$h\backslash c$	0	1
3	$\Omega^3 k$	$\Omega^3\left(\mathrm{Sym}^2 E\right)$
2	$\Omega^2 k$	$\Omega^2\left(\mathrm{Sym}^2 E\right)$
1	Ωk	$\Omega\left(\mathrm{Sym}^2 E\right)$
0	k	$\mathrm{Sym}^2 E$

0	1
$\Omega^3\left(\mathrm{Sym}^3 E\right)$	$\Omega^3 E$
$\Omega^2\left(\mathrm{Sym}^3 E\right)$	$\Omega^2 E$
$\Omega\left(\mathrm{Sym}^3 E\right)$	ΩE
$\mathrm{Sym}^3 E$	E

Let us construct two $(p-1)\times(p-1)/2$ tables (for $p=5$). Label rows by $h\in\mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0\leq c\leq(p-3)/2$. We fill the first table with $\Omega^h\left(\mathrm{Sym}^{2c} E\right)$ and the second one with $\Omega^h\left(\mathrm{Sym}^{p-2-2c} E\right)$.

Diagrammatic representation ring

$h \backslash c$	0	1
3	$\Omega^3 k$	$\Omega^3 \left(\mathrm{Sym}^2 E\right)$
2	$\Omega^2 k$	$\Omega^2 \left(\mathrm{Sym}^2 E\right)$
1	Ωk	$\Omega \left(\mathrm{Sym}^2 E\right)$
0	k	$\mathrm{Sym}^2 E$

0	1
$\Omega^3 \left(\mathrm{Sym}^3 E\right)$	$\Omega^3 E$
$\Omega^2 \left(\mathrm{Sym}^3 E\right)$	$\Omega^2 E$
$\Omega \left(\mathrm{Sym}^3 E\right)$	ΩE
$\mathrm{Sym}^3 E$	E

Let us construct two $(p-1) \times (p-1)/2$ tables (for $p = 5$). Label rows by $h \in \mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0 \leq c \leq (p-3)/2$. We fill the first table with $\Omega^h \left(\mathrm{Sym}^{2c} E\right)$ and the second one with $\Omega^h \left(\mathrm{Sym}^{p-2-2c} E\right)$. Then multiplication by X, respectively, Y corresponds to going up, respectively, changing a table.

Diagrammatic representation ring

$h\backslash c$	0	1
3	$\Omega^3 k$	$\Omega^3\left(\mathrm{Sym}^2 E\right)$
2	$\Omega^2 k$	$\Omega^2\left(\mathrm{Sym}^2 E\right)$
1	Ωk	$\Omega\left(\mathrm{Sym}^2 E\right)$
0	k	$\mathrm{Sym}^2 E$

0	1
$\Omega^3\left(\mathrm{Sym}^3 E\right)$	$\Omega^3 E$
$\Omega^2\left(\mathrm{Sym}^3 E\right)$	$\Omega^2 E$
$\Omega\left(\mathrm{Sym}^3 E\right)$	ΩE
$\mathrm{Sym}^3 E$	E

Let us construct two $(p-1)\times(p-1)/2$ tables (for $p=5$). Label rows by $h\in\mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0\leq c\leq(p-3)/2$. We fill the first table with $\Omega^h\left(\mathrm{Sym}^{2c} E\right)$ and the second one with $\Omega^h\left(\mathrm{Sym}^{p-2-2c} E\right)$. Then multiplication by X, respectively, Y corresponds to going up, respectively, changing a table.

Example: $\Omega\left(\mathrm{Sym}^3 E\right)\otimes\Omega^2\left(\mathrm{Sym}^3 E\right)\overset{p}{\cong}\Omega^3\left(\mathrm{Sym}^0 E\right)$ can be written using Ψ as $XYX^2Y=X^3$.

Diagrammatic representation ring

$h\backslash c$	0	1
3	$\Omega^3 k$	$\Omega^3\left(\mathrm{Sym}^2 E\right)$
2	$\Omega^2 k$	$\Omega^2\left(\mathrm{Sym}^2 E\right)$
1	Ωk	$\Omega\left(\mathrm{Sym}^2 E\right)$
0	k	$\mathrm{Sym}^2 E$

$h\backslash c$	0	1
3	$\Omega^3\left(\mathrm{Sym}^3 E\right)$	$\Omega^3 E$
2	$\Omega^2\left(\mathrm{Sym}^3 E\right)$	$\Omega^2 E$
1	$\Omega\left(\mathrm{Sym}^3 E\right)$	ΩE
0	$\mathrm{Sym}^3 E$	E

Let us construct two $(p-1)\times(p-1)/2$ tables (for $p=5$). Label rows by $h\in\mathbb{Z}/(p-1)\mathbb{Z}$ and columns by $0\le c\le(p-3)/2$. We fill the first table with $\Omega^h\left(\mathrm{Sym}^{2c} E\right)$ and the second one with $\Omega^h\left(\mathrm{Sym}^{p-2-2c} E\right)$. Then multiplication by X, respectively, Y corresponds to going up, respectively, changing a table.

Example: $\Omega\left(\mathrm{Sym}^3 E\right)\otimes\Omega^2\left(\mathrm{Sym}^3 E\right)\overset{p}{\cong}\Omega^3\left(\mathrm{Sym}^0 E\right)$ can be written using Ψ as $XYX^2Y=X^3$. Alternatively, using the table, this is $k\otimes k$ with switching tables twice and going up three times.

Outline

p-λ-rings

By restricting the inverse of the isomorphism
$\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains
$\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$.

p-λ-rings

By restricting the inverse of the isomorphism
$\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains
$\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. Letting $Y = -1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.

p-λ-rings

By restricting the inverse of the isomorphism
$\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains
$\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. Letting $Y = -1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.
We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^{l}$.

p-λ-rings

By restricting the inverse of the isomorphism $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains $\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. Letting $Y = -1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.

We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^l$.

Proposition

Θ is a homomorphism of p-λ-rings.

p-λ-rings

By restricting the inverse of the isomorphism $\Psi\colon \mathbb{Z}[\zeta_p+\zeta_p^{-1}][X,Y]/(X^{p-1}-1, Y^2-1) \to \overline{R(G)}$ one obtains $\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p+\zeta_p^{-1}][Y]/(Y^2-1)$. Letting $Y=-1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p+\zeta_p^{-1}]$.

We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^{l}$.

Proposition

Θ is a homomorphism of p-λ-rings.

p-λ-rings come with operations $\{\nu\}$ labelled by p-small partitions ν.

p-λ-rings

By restricting the inverse of the isomorphism $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains $\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. Letting $Y = -1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.

We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^{l}$.

Proposition

Θ is a homomorphism of p-λ-rings.

p-λ-rings come with operations $\{\nu\}$ labelled by p-small partitions ν.

For R_I the operations are given by Schur functors. That is, $\{\nu\}\,\overline{V} = \overline{\nabla^\nu V}$.

p-λ-rings

By restricting the inverse of the isomorphism $\Psi\colon \mathbb{Z}[\zeta_p + \zeta_p^{-1}][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to \overline{R(G)}$ one obtains $\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}][Y]/(Y^2 - 1)$. Letting $Y = -1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.

We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^{l}$.

Proposition

Θ is a homomorphism of p-λ-rings.

p-λ-rings come with operations $\{\nu\}$ labelled by p-small partitions ν.

For R_I the operations are given by Schur functors. That is, $\{\nu\}\,\overline{V} = \overline{\nabla^\nu V}$.

For $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ the operations correspond to Schur functions.

p-λ-rings

By restricting the inverse of the isomorphism
$\Psi\colon \mathbb{Z}[\zeta_p+\zeta_p^{-1}][X,Y]/(X^{p-1}-1, Y^2-1) \to \overline{R(G)}$ one obtains
$\Psi_1\colon R_I \to \mathbb{Z}[\zeta_p+\zeta_p^{-1}][Y]/(Y^2-1)$. Letting $Y=-1$ we get a surjective ring homomorphism $\Theta\colon R_I \to \mathbb{Z}[\zeta_p+\zeta_p^{-1}]$.
We have $\Theta(\overline{\mathrm{Sym}^l E}) = \zeta_p^{-l} + \zeta_p^{-l+2} + \cdots + \zeta_p^{l}$.

Proposition

Θ is a homomorphism of p-λ-rings.

p-λ-rings come with operations $\{\nu\}$ labelled by p-small partitions ν.
For R_I the operations are given by Schur functors. That is, $\{\nu\}\,\overline{V} = \overline{\nabla^\nu V}$.
For $\mathbb{Z}[\zeta_p+\zeta_p^{-1}]$ the operations correspond to Schur functions.
Example: Let $f=\zeta_p^2+1+\zeta_p^{-2}$. Then
$\{(2,1)\}\,f = s_{2,1}(\zeta_p^2,1,\zeta_p^{-2}) = \zeta_p^4 + 2\zeta_p^2 + 2 + 2\zeta_p^{-2} + \zeta_p^{-4}$.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^\nu \operatorname{Sym}^l E}) = s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^{l})$.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^\nu \operatorname{Sym}^l E}) = s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^{l})$.

The only obstacle left is that Θ is not invertible.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^\nu \operatorname{Sym}^l E}) = s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^l)$.

The only obstacle left is that Θ is not invertible. However, it can be shown that one obtains $\nabla^\nu \operatorname{Sym}^l E$ in the stable category by taking the 'least element' in the preimage of $s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^l)$.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^\nu \operatorname{Sym}^l E}) = s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^l)$.

The only obstacle left is that Θ is not invertible. However, it can be shown that one obtains $\nabla^\nu \operatorname{Sym}^l E$ in the stable category by taking the 'least element' in the preimage of $s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^l)$.

Example: We know that $\Theta(\overline{\nabla^{(2,1)} \operatorname{Sym}^2 E}) = s_{(2,1)}(\zeta_p^{-2}, 1, \zeta_p^2) = \zeta_p^4 + 2\zeta_p^2 + 2 + 2\zeta_p^{-2} + \zeta_p^{-4} = (\zeta_p^4 + \zeta_p^2 + 1 + \zeta_p^{-2} + \zeta_p^{-4}) + (\zeta_p^2 + 1 + \zeta_p^{-2})$.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^\nu \operatorname{Sym}^l E}) = s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \dots, \zeta_p^l)$.

The only obstacle left is that Θ is not invertible. However, it can be shown that one obtains $\nabla^\nu \operatorname{Sym}^l E$ in the stable category by taking the 'least element' in the preimage of $s_\nu(\zeta_p^{-l}, \zeta_p^{-l+2}, \dots, \zeta_p^l)$.

Example: We know that $\Theta(\overline{\nabla^{(2,1)} \operatorname{Sym}^2 E}) = s_{(2,1)}(\zeta_p^{-2}, 1, \zeta_p^2) = \zeta_p^4 + 2\zeta_p^2 + 2 + 2\zeta_p^{-2} + \zeta_p^{-4} = (\zeta_p^4 + \zeta_p^2 + 1 + \zeta_p^{-2} + \zeta_p^{-4}) + (\zeta_p^2 + 1 + \zeta_p^{-2})$.

If $p > 5$, then $\nabla^{(2,1)} \operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^4 E \oplus \operatorname{Sym}^2 E$.

Finding modular plethysms I

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then $\Theta(\overline{\nabla^{\nu} \operatorname{Sym}^{l} E}) = s_{\nu}(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^{l})$.

The only obstacle left is that Θ is not invertible. However, it can be shown that one obtains $\nabla^{\nu} \operatorname{Sym}^{l} E$ in the stable category by taking the 'least element' in the preimage of $s_{\nu}(\zeta_p^{-l}, \zeta_p^{-l+2}, \ldots, \zeta_p^{l})$.

Example: We know that $\Theta(\overline{\nabla^{(2,1)} \operatorname{Sym}^{2} E}) = s_{(2,1)}(\zeta_p^{-2}, 1, \zeta_p^{2}) = \zeta_p^4 + 2\zeta_p^2 + 2 + 2\zeta_p^{-2} + \zeta_p^{-4} = (\zeta_p^4 + \zeta_p^2 + 1 + \zeta_p^{-2} + \zeta_p^{-4}) + (\zeta_p^2 + 1 + \zeta_p^{-2})$.

If $p > 5$, then $\nabla^{(2,1)} \operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^4 E \oplus \operatorname{Sym}^2 E$.

If $p = 5$, then $\nabla^{(2,1)} \operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^2 E$.

Stanley's Hook Content Formula

Fix a partition λ and let $(i, j) \in Y(\lambda)$. The hook length of (i, j) is $\lambda_i + \lambda'_j - i - j + 1$.

Stanley's Hook Content Formula

Fix a partition λ and let $(i, j) \in Y(\lambda)$. The hook length of (i, j) is $\lambda_i + \lambda'_j - i - j + 1$. The content of (i, j) equals $j - i$.

Stanley's Hook Content Formula

Fix a partition λ and let $(i, j) \in Y(\lambda)$. The hook length of (i, j) is $\lambda_i + \lambda'_j - i - j + 1$. The content of (i, j) equals $j - i$. We denote by $\mathcal{H}$ the multiset of hook lengths of λ, by $\mathcal{C}$ the multiset of all contents and by $\mathcal{C}_s$ the multiset of shifted contents by s (obtained by adding s to all contents).

Stanley's Hook Content Formula

Fix a partition λ and let $(i,j) \in Y(\lambda)$. The hook length of (i,j) is $\lambda_i + \lambda'_j - i - j + 1$. The content of (i,j) equals $j - i$. We denote by $\mathcal{H}$ the multiset of hook lengths of λ, by $\mathcal{C}$ the multiset of all contents and by $\mathcal{C}_s$ the multiset of shifted contents by s (obtained by adding s to all contents).

Example: Hook lengths, contents and shifted contents by 3 of the partition $(3,2)$.

4	3	1
2	1	

0	1	2
-1	0	

3	4	5
2	3	

Stanley's Hook Content Formula

Fix a partition λ and let $(i,j) \in Y(\lambda)$. The hook length of (i,j) is $\lambda_i + \lambda'_j - i - j + 1$. The content of (i,j) equals $j - i$. We denote by $\mathcal{H}$ the multiset of hook lengths of λ, by $\mathcal{C}$ the multiset of all contents and by $\mathcal{C}_s$ the multiset of shifted contents by s (obtained by adding s to all contents).

Example: Hook lengths, contents and shifted contents by 3 of the partition $(3,2)$.

4	3	1
2	1	

0	1	2
-1	0	

3	4	5
2	3	

Theorem (Stanley's Hook Content Formula)

Let λ be a partition, l a non-negative integer and q a variable. Then

$$s_\lambda(q^{-l}, q^{-l+2}, \ldots, q^{l}) = \frac{\prod_{c \in \mathcal{C}_{l+1}} (q^c - q^{-c})}{\prod_{h \in \mathcal{H}} (q^h - q^{-h})}.$$

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^\nu \operatorname{Sym}^l E}\right) = \frac{\prod_{c \in \mathcal{C}_{l+1}}(\zeta_p^c - \zeta_p^{-c})}{\prod_{h \in \mathcal{H}}(\zeta_p^h - \zeta_p^{-h})}.$$

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu}\operatorname{Sym}^{l} E}\right) = \frac{\prod_{c\in\mathcal{C}_{l+1}}(\zeta_p^{c} - \zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h} - \zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)}\operatorname{Sym}^2 E$. Recall that $\mathcal{H} = \{1, 1, 2, 3, 4\}$ and $\mathcal{C}_3 = \{2, 3, 3, 4, 5\}$.

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu}\operatorname{Sym}^{l} E}\right) = \frac{\prod_{c\in\mathcal{C}_{l+1}}(\zeta_p^{c} - \zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h} - \zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)}\operatorname{Sym}^2 E$. Recall that $\mathcal{H} = \{1,1,2,3,4\}$ and $\mathcal{C}_3 = \{2,3,3,4,5\}$. $\Theta\left(\overline{\nabla^{(3,2)}\operatorname{Sym}^2 E}\right) = \frac{\prod_{c\in\mathcal{C}_3}(\zeta_p^{c}-\zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h}-\zeta_p^{-h})}$

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu}\operatorname{Sym}^{l} E}\right) = \frac{\prod_{c\in\mathcal{C}_{l+1}}(\zeta_p^{c}-\zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h}-\zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)}\operatorname{Sym}^2 E$. Recall that $\mathcal{H} = \{1, 1, 2, 3, 4\}$ and $\mathcal{C}_3 = \{2, 3, 3, 4, 5\}$. $\Theta\left(\overline{\nabla^{(3,2)}\operatorname{Sym}^2 E}\right) = \frac{\prod_{c\in\mathcal{C}_3}(\zeta_p^{c}-\zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h}-\zeta_p^{-h})}$

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu}\operatorname{Sym}^{l} E}\right) = \frac{\prod_{c\in\mathcal{C}_{l+1}}(\zeta_p^{c} - \zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h} - \zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)}\operatorname{Sym}^2 E$. Recall that $\mathcal{H} = \{1, 1, 2, 3, 4\}$ and $\mathcal{C}_3 = \{2, 3, 3, 4, 5\}$. $\Theta\left(\overline{\nabla^{(3,2)}\operatorname{Sym}^2 E}\right) = \frac{\prod_{c\in\mathcal{C}_3}(\zeta_p^c - \zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^h - \zeta_p^{-h})} = \frac{\zeta_p^5-\zeta_p^{-5}}{\zeta_p-\zeta_p^{-1}}\frac{\zeta_p^3-\zeta_p^{-3}}{\zeta_p-\zeta_p^{-1}} = \zeta_p^6 + 2\zeta_p^4 + 3\zeta_p^2 + 3 + 3\zeta_p^{-2} + 2\zeta_p^{-4} + \zeta_p^{-6}$.

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \leq l \leq p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu} \operatorname{Sym}^{l} E}\right) = \frac{\prod_{c \in \mathcal{C}_{l+1}} (\zeta_p^{c} - \zeta_p^{-c})}{\prod_{h \in \mathcal{H}} (\zeta_p^{h} - \zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)} \operatorname{Sym}^2 E$. Recall that $\mathcal{H} = \{1,1,2,3,4\}$ and $\mathcal{C}_3 = \{2,3,3,4,5\}$. $\Theta\left(\overline{\nabla^{(3,2)} \operatorname{Sym}^2 E}\right) = \frac{\prod_{c \in \mathcal{C}_3} (\zeta_p^{c} - \zeta_p^{-c})}{\prod_{h \in \mathcal{H}} (\zeta_p^{h} - \zeta_p^{-h})} =$
$\frac{\zeta_p^5 - \zeta_p^{-5}}{\zeta_p - \zeta_p^{-1}} \frac{\zeta_p^3 - \zeta_p^{-3}}{\zeta_p - \zeta_p^{-1}} = \zeta_p^6 + 2\zeta_p^4 + 3\zeta_p^2 + 3 + 3\zeta_p^{-2} + 2\zeta_p^{-4} + \zeta_p^{-6}$.
If $p > 7$ we can conclude that
$\nabla^{(3,2)} \operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^6 E \oplus \operatorname{Sym}^4 E \oplus \operatorname{Sym}^2 E$.

Finding modular plethysms II

Corollary

Let ν be a p-small partition and $0 \le l \le p-2$. Then we can write

$$\Theta\left(\overline{\nabla^{\nu}\operatorname{Sym}^{l} E}\right) = \frac{\prod_{c\in\mathcal{C}_{l+1}}(\zeta_p^{c}-\zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h}-\zeta_p^{-h})}.$$

Example: Let us compute $\nabla^{(3,2)}\operatorname{Sym}^2 E$. Recall that $\mathcal{H}=\{1,1,2,3,4\}$ and $\mathcal{C}_3=\{2,3,3,4,5\}$. $\Theta\left(\overline{\nabla^{(3,2)}\operatorname{Sym}^2 E}\right)=\frac{\prod_{c\in\mathcal{C}_3}(\zeta_p^{c}-\zeta_p^{-c})}{\prod_{h\in\mathcal{H}}(\zeta_p^{h}-\zeta_p^{-h})}=$
$\frac{\zeta_p^5-\zeta_p^{-5}}{\zeta_p-\zeta_p^{-1}}\frac{\zeta_p^3-\zeta_p^{-3}}{\zeta_p-\zeta_p^{-1}}=\zeta_p^6+2\zeta_p^4+3\zeta_p^2+3+3\zeta_p^{-2}+2\zeta_p^{-4}+\zeta_p^{-6}$.
If $p>7$ we can conclude that
$\nabla^{(3,2)}\operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^6 E\oplus\operatorname{Sym}^4 E\oplus\operatorname{Sym}^2 E$. If $p=7$, then
$\nabla^{(3,2)}\operatorname{Sym}^2 E \overset{p}{\cong} \operatorname{Sym}^4 E\oplus\operatorname{Sym}^2 E$.

Main results I

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition. Then $\nabla^\nu \operatorname{Sym}^l E$ is projective if and only if $\nu_1 \geq p-l$ or $\ell(\nu) \geq l+2$.

Main results I

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition. Then $\nabla^\nu \operatorname{Sym}^l E$ is projective if and only if $\nu_1 \geq p - l$ or $\ell(\nu) \geq l+2$.

Note that the second inequality $\ell(\nu) \geq l+2$ is equivalent to $\nabla^\nu \operatorname{Sym}^l E$ being the zero module.

Main results I

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition. Then $\nabla^{\nu} \operatorname{Sym}^{l} E$ is projective if and only if $\nu_1 \geq p-l$ or $\ell(\nu) \geq l+2$.

Note that the second inequality $\ell(\nu) \geq l+2$ is equivalent to $\nabla^{\nu} \operatorname{Sym}^{l} E$ being the zero module.

In 2021 Paget and Wildon classified all irreducible plethysms of the natural $\mathbb{C}SL_2(\mathbb{C})$-module. We can show an analogous result in prime characteristic.

Main results I

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition. Then $\nabla^\nu \operatorname{Sym}^l E$ is projective if and only if $\nu_1 \geq p-l$ or $\ell(\nu) \geq l+2$.

Note that the second inequality $\ell(\nu) \geq l+2$ is equivalent to $\nabla^\nu \operatorname{Sym}^l E$ being the zero module.

In 2021 Paget and Wildon classified all irreducible plethysms of the natural $\mathbb{C}\mathrm{SL}_2(\mathbb{C})$-module. We can show an analogous result in prime characteristic.

We say that a kG-module is stably-irreducible if it has only one non-projective indecomposable summand which is moreover irreducible.

Main results II

Theorem (T, 2022)

Let $0 \le l \le p-2$ and let ν be a p-small partition with $\ell(\nu) \le l$ and $\nu_1 \le p-l-2$. Then $\nabla^\nu \operatorname{Sym}^l E$ is stably-irreducible if and only if (at least) one of the following happens:

1. (elementary cases) $\nu = \varnothing$ or $\nu = (1)$,
2. (row cases) $\nu = (p-l-2)$ or $l = 1$,
3. (column cases) $\nu = (1^l)$ or $l = p-3$,
4. (rectangular cases) $p = 7$ and either $\nu = (2,2,2)$ with $l = 3$ or $\nu = (3,3)$ with $l = 2$.

Main results II

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition with $\ell(\nu) \leq l$ and $\nu_1 \leq p-l-2$. Then $\nabla^\nu \operatorname{Sym}^l E$ is stably-irreducible if and only if (at least) one of the following happens:

1. (elementary cases) $\nu = \varnothing$ or $\nu = (1)$,
2. (row cases) $\nu = (p-l-2)$ or $l = 1$,
3. (column cases) $\nu = (1^l)$ or $l = p-3$,
4. (rectangular cases) $p = 7$ and either $\nu = (2,2,2)$ with $l = 3$ or $\nu = (3,3)$ with $l = 2$.

One can drop the two constraints for the p-small partition by adding rows of length $p-l-1$ and columns of length $l+1$.

Main results II

Theorem (T, 2022)

Let $0 \leq l \leq p-2$ and let ν be a p-small partition with $\ell(\nu) \leq l$ and $\nu_1 \leq p-l-2$. Then $\nabla^\nu \operatorname{Sym}^l E$ is stably-irreducible if and only if (at least) one of the following happens:

1. (elementary cases) $\nu = \varnothing$ or $\nu = (1)$,
2. (row cases) $\nu = (p-l-2)$ or $l = 1$,
3. (column cases) $\nu = (1^l)$ or $l = p-3$,
4. (rectangular cases) $p = 7$ and either $\nu = (2,2,2)$ with $l = 3$ or $\nu = (3,3)$ with $l = 2$.

One can drop the two constraints for the p-small partition by adding rows of length $p-l-1$ and columns of length $l+1$. Note that the above list of (ν, l) is closed under the involution $(\nu, l) \mapsto (\nu', p-l-2)$.

Outline

Endotrivial modules

Recall that a kH-module V is endotrivial if there is a kH-module W such that $V \otimes W \overset{p}{\cong} k$.

Endotrivial modules

Recall that a kH-module V is endotrivial if there is a kH-module W such that $V \otimes W \overset{p}{\cong} k$.

Examples: If k denotes the trivial kH-module, then Ωk is endotrivial.

Endotrivial modules

Recall that a kH-module V is endotrivial if there is a kH-module W such that $V \otimes W \overset{p}{\cong} k$.

Examples: If k denotes the trivial kH-module, then Ωk is endotrivial. The irreducible kG-module $\mathrm{Sym}^{p-2} E$ is endotrivial since the Clebsch–Gordan rule gives $\mathrm{Sym}^{p-2} E \otimes \mathrm{Sym}^{p-2} E \overset{p}{\cong} k$.

Endotrivial modules

Recall that a kH-module V is endotrivial if there is a kH-module W such that $V \otimes W \overset{p}{\cong} k$.

Examples: If k denotes the trivial kH-module, then Ωk is endotrivial. The irreducible kG-module $\mathrm{Sym}^{p-2} E$ is endotrivial since the Clebsch–Gordan rule gives $\mathrm{Sym}^{p-2} E \otimes \mathrm{Sym}^{p-2} E \overset{p}{\cong} k$.

An important property of endotrivial modules is that a tensor product of an endotrivial module and a non-projective indecomposable module has precisely one non-projective indecomposable summand.

Endotrivial modules

Recall that a kH-module V is endotrivial if there is a kH-module W such that $V \otimes W \overset{p}{\cong} k$.

Examples: If k denotes the trivial kH-module, then Ωk is endotrivial. The irreducible kG-module $\mathrm{Sym}^{p-2} E$ is endotrivial since the Clebsch–Gordan rule gives $\mathrm{Sym}^{p-2} E \otimes \mathrm{Sym}^{p-2} E \overset{p}{\cong} k$.

An important property of endotrivial modules is that a tensor product of an endotrivial module and a non-projective indecomposable module has precisely one non-projective indecomposable summand.

Example: For any $0 \leq i \leq p-2$ we have
$\mathrm{Sym}^i E \otimes \mathrm{Sym}^{p-2} E \overset{p}{\cong} \mathrm{Sym}^{p-2-i} E$.

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,
- this n-fold tensor product has a unique non-projective summand,

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,
- this n-fold tensor product has a unique non-projective summand,
- by dimension counting this summand is $\mathrm{Sym}^n V$,

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^\nu(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^\nu W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,
- this n-fold tensor product has a unique non-projective summand,
- by dimension counting this summand is $\operatorname{Sym}^n V$,
- for any $\lambda \vdash n$ the module $\nabla^\lambda V$ is projective unless $\lambda = (n)$,

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv 1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,
- this n-fold tensor product has a unique non-projective summand,
- by dimension counting this summand is $\mathrm{Sym}^n V$,
- for any $\lambda \vdash n$ the module $\nabla^{\lambda} V$ is projective unless $\lambda = (n)$,
- expand $\nabla^{\nu}(V \otimes W)$ using the Kronecker coefficients.

Exchange lemma

Proposition (T, 2022)

Let ν be a p-small partition of n, V an endotrivial kH-module and W any kH-module. If d is the dimension of V, then

$$\nabla^{\nu}(V \otimes W) \overset{p}{\cong} \begin{cases} V^{\otimes n} \otimes \nabla^{\nu} W & \text{if } d \equiv 1 \bmod p, \\ V^{\otimes n} \otimes \nabla^{\nu'} W & \text{if } d \equiv -1 \bmod p. \end{cases}$$

Steps of the proof for $d \equiv -1 \bmod p$ (throughout n equals the size of ν):

- expand $V^{\otimes n}$ using Schur–Weyl duality,
- this n-fold tensor product has a unique non-projective summand,
- by dimension counting this summand is $\bigwedge^n V$,
- for any $\lambda \vdash n$ the module $\nabla^{\lambda} V$ is projective unless $\lambda = (1^n)$,
- expand $\nabla^{\nu}(V \otimes W)$ using the Kronecker coefficients.

Corollaries of the exchange lemma

Corollary

If W is a kH-module and $\nu \vdash n$ with $n < p$, then $\nabla^{\nu}(\Omega W) \overset{p}{\cong} \Omega^{n}(\nabla^{\nu'} W)$.

Corollaries of the exchange lemma

Corollary

If W is a kH-module and $\nu \vdash n$ with $n < p$, then $\nabla^\nu(\Omega W) \overset{p}{\cong} \Omega^n(\nabla^{\nu'} W)$.

This helps generalise the earlier classifications of certain plethysms $\nabla^\nu \operatorname{Sym}^l E$ by replacing $\operatorname{Sym}^l E$ by any indecomposable kG-module.

Corollaries of the exchange lemma

Corollary

If W is a kH-module and $\nu \vdash n$ with $n < p$, then $\nabla^\nu(\Omega W) \overset{p}{\cong} \Omega^n(\nabla^{\nu'} W)$.

This helps generalise the earlier classifications of certain plethysms $\nabla^\nu \operatorname{Sym}^l E$ by replacing $\operatorname{Sym}^l E$ by any indecomposable kG-module.

Corollary

For $0 \leq l \leq p-2$ and $\nu \vdash n$ with $n < p$ we have

$$\nabla^\nu \operatorname{Sym}^{p-2-l} E \overset{p}{\cong} \begin{cases} \nabla^{\nu'} \operatorname{Sym}^l E & \text{if } n \text{ is even,} \\ \operatorname{Sym}^{p-2} E \otimes \nabla^{\nu'} \operatorname{Sym}^l E & \text{if } n \text{ is odd.} \end{cases}$$

Corollaries of the exchange lemma

Corollary

If W is a kH-module and $\nu \vdash n$ with $n < p$, then $\nabla^{\nu}(\Omega W) \overset{p}{\cong} \Omega^{n}(\nabla^{\nu'} W)$.

This helps generalise the earlier classifications of certain plethysms $\nabla^{\nu} \operatorname{Sym}^{l} E$ by replacing $\operatorname{Sym}^{l} E$ by any indecomposable kG-module.

Corollary

For $0 \leq l \leq p-2$ and $\nu \vdash n$ with $n < p$ we have

$$\nabla^{\nu} \operatorname{Sym}^{p-2-l} E \overset{p}{\cong} \begin{cases} \nabla^{\nu'} \operatorname{Sym}^{l} E & \text{if } n \text{ is even,} \\ \operatorname{Sym}^{p-2} E \otimes \nabla^{\nu'} \operatorname{Sym}^{l} E & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the number of the non-projective indecomposable summands of $\nabla^{\nu} \operatorname{Sym}^{l} E$ is invariant under the involution $(\nu, l) \mapsto (\nu', p-l-2)$.