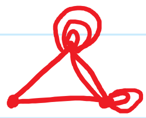
HOMFLYPT, LMOV, DT

Quivers: Q: $I = \{i, j, \ldots\}$ $\quad E = \{\alpha : i \to j\}$

Q-reps
- f.d. $\mathbb{C}$-vectorspace V_i for $i \in I$
- linear map $f_\alpha : V_i \to V_j$ for $\alpha : i \to j$

Fix a dimension vector $(\dim V_i)_{i \in I} = (d_i)_{i \in I} = \underline{d}$

and set:

$$M_{\underline{d}} := \prod_{i,j} \mathbb{C}^{d_i d_j \cdot \#\{\alpha : i \to j\}} \circlearrowleft G_{\underline{d}} = \prod_{i \in I} GL(d_i).$$

Can compute $\mathcal{H}_{\underline{d}} = H^*_{G_{\underline{d}}}(M_{\underline{d}}, \mathbb{Q}) \quad (\cong H^*(BG_{\underline{d}}, \mathbb{Q}))$

Kontsevich-Soibelman:

$\mathcal{H} = \bigoplus_{\underline{d} \in \mathbb{N}^I} \mathcal{H}_{\underline{d}}$ admits an associative multiplication

$\mathcal{H}$ is called the (rational) cohomological Hall algebra of Q

Efimov: if Q is symmetric $(\#\{\alpha : i \to j\} = \#\{\alpha : j \to i\})$,

then $\mathcal{H}$ is $\mathbb{N}^I \times \mathbb{Z}$-graded & free supercommutative, (after sign twist)

generated by some $V = V^{prim} \otimes \mathbb{C}[x]$ $\deg x = (\underline{0}, 2)$.

The motivic DT-series of Q, in variables $x_1, \ldots, x_p, q$ where $p = |I|$

$$P_Q(\underline{x}) := \sum_{\substack{\underline{d} \in \mathbb{N}^p \\ k \in \mathbb{Z}}} (-q)^{-k} x_1^{d_1} \cdots x_p^{d_p} \dim(\mathcal{H}_{\underline{d}k}) = \sum_{\underline{d} \in \mathbb{N}^p} \frac{(-q)^{-\chi_Q(\underline{d}, \underline{d})} x_1^{d_1} \cdots x_p^{d_p}}{(q^{-2}; q^{-2})_{d_1} \cdots (q^{-2}; q^{-2})_{d_p}}$$

where $\chi_Q(\underline{d}, \underline{e}) = \underline{d}(I - Q)\underline{e}^t$, $\quad (x, y)_z = \prod_{j=0}^{z-1} (1 - xy^j)$,

Euler form $\quad$ Pochhammer symb.

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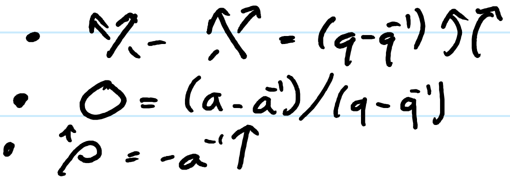
Euler form Pochhammer symb.

The HOMFLYPT "polynomial" $\tilde{P}_1 : \{\text{framed oriented links}\} \longrightarrow \mathbb{Q}(q)[a^{\pm 1}] =: A$

is determined by skein relations :



reduced HOMFLYPT:

$$P_1(L) := \frac{\tilde{P}_1(L)}{\tilde{P}_1(\bigcirc)}$$

Fact: P_1 is the large N limit of $U_q(\mathfrak{gl}_N)$ RT invariants $a=q^N$
for $j \geq 1$ also have P_j $\Lambda^j \mathbb{C}_q^N$-colored

Conjecture A (Kucharski–Reineke–Stošić–Sułkowski, modified)
For a large class of links L with c components, there exist symmetric quivers Q_L and $\underline{a}, \underline{s} \in \mathbb{Z}^p$ (p = # vertices in Q_L) s.t.:

$$P(L) := \sum_{j \in \mathbb{N}} P_j(L)(q^2,q^2)_j^{c-2} x^j = P_{Q_L}\Big|_{x_i \mapsto (-q)^{s_i} a^{a_i} x} \in A[[x]]$$

Conjecture B (KRSS) For a large class of knots K, there are quivers Q_K such that:

$$P(K) = \sum_{j \in \mathbb{N}} \frac{P_j(K)}{(q^2,q^2)_j} x^j = P_{Q_K}\Big|_{\dots}$$

and $\{\text{vertices}\} \overset{1:1}{\longleftrightarrow}$ generators of $HHH(K)$
loops = homological degree

Remarks
- checked by KRSS for knots with ≤ 6 crossings, 2-strand torus knots and twist knots.
- maybe stronger than previous structural predictions (q-holonomicity, LMOV integrality)

e.g.:
$$P_j(K) = \sum_{d_1+\dots+d_p=j} (-q)^{\underline{s}\cdot\underline{d}} \cdot q^{\underline{d}^T \cdot Q_K \cdot \underline{d}} \cdot a^{\underline{a}\cdot\underline{d}} \cdot \begin{bmatrix} j \\ d_1, d_2, \dots, d_p \end{bmatrix}_q$$

$d_1+\dots+d_p=j$

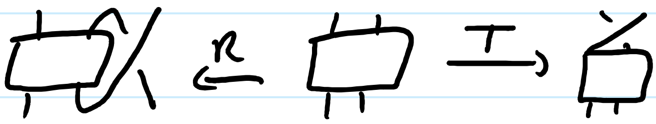
- Origin: $SU(N)$ CS theory on T^*S^3 $\rightsquigarrow$ top string theory (GW/DT) on resolved conifold (Ekholm-Kucharski-Longhi)

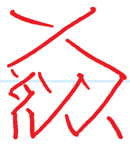
- the large-color behavior of quantum invts. conj. captures info about $S^3 \setminus n(K)$:
  - hyperbolic volume (volume conj.)
  - ∂-slopes and topology of incompressible surfaces (slope conj.)
  - $SL(2,\mathbb{C})$ character variety and A-polynomial (AJ-conj.)
  - knot contact homology

Thm (Stošić-W, 17) Conj. A and B hold for rational links.

Thm (Stošić-W, 20) Conj. A holds for arborescent links.

Rational tangles: built from $\uparrow\uparrow$ by operations

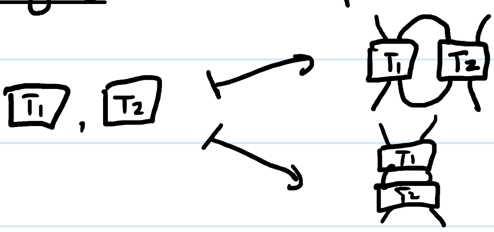


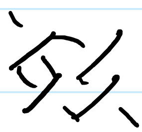


$$\mathbb{Q} \ni p/q = a_r + \cfrac{1}{a_{r-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}} \mapsto T^{a_r} R^{a_{r-1}} \cdots R^{a_2} T^{a_1} (\uparrow\uparrow)$$

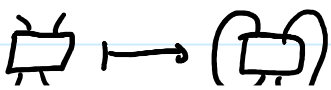
r odd

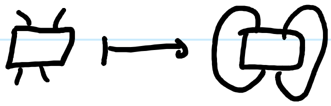
Arborescent tangles: closure of rational tangles under gluing



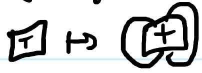


Rational/arborescent Links:

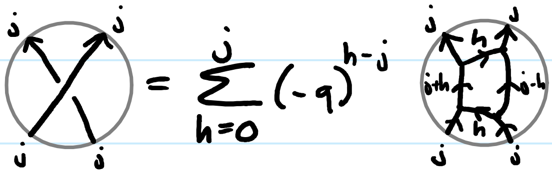
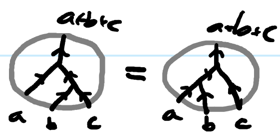
obtained by closure 

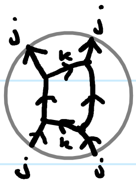
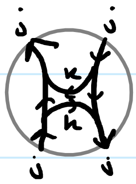
obtained by closure 

Proof strategy : Divide & conquer:

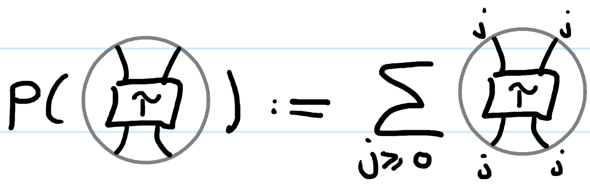
1) notion of quiver for 4-ended tangles
2) tangles  have quivers
3) gluing formulas for quivers 
4) closing formulas 

0) HOMFLYPT skein theory for colored tangles

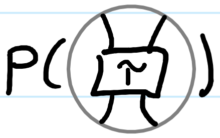
- 
- Local, linear relations on webs, e.g.: 

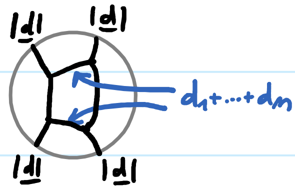
Bases:  ,  $0 \leq h \leq j$

UP[j,k] OP[j,k]

1) Definition: 

this is said to be in quiver form if
$\exists m,n \geq 0$, $\underline{a}, \underline{s} \in \mathbb{Z}^{m+n}$, $Q \in Mat_{(m+n)\times(m+n)}(\mathbb{Z})$ s.t.



$$= \sum_{\underline{d} \in \mathbb{N}^{m+n}} (-q)^{\underline{s}\cdot\underline{d}} \cdot q^{\underline{d}^t \cdot Q \cdot \underline{d}} \cdot a^{\underline{a}\cdot\underline{d}} \cdot \begin{bmatrix} d_1+\dots+d_m \\ d_1,\dots,d_m \end{bmatrix}_q \begin{bmatrix} d_{m+1}+\dots+d_{m+n} \\ d_{m+1},\dots,d_{m+n} \end{bmatrix}_q \cdot (q^2,q^2)^{-c} \cdot$$ 

where $c = \#$ closed cpts of $\tilde{T}$.

2) Examples

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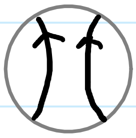
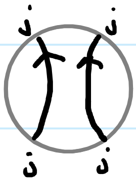
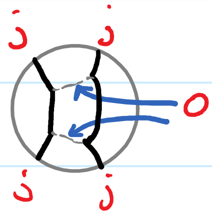
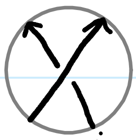
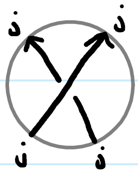
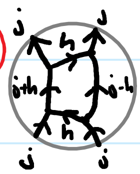
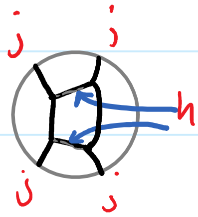
$P($  $) := \sum_{j \geq 0}$ 

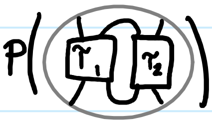
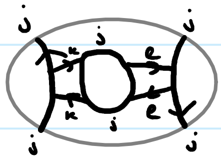
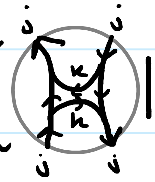
$= \sum_{\substack{d \in \mathbb{N}^{0+1} \\ = (j)}} (-q)^{j \cdot 0} \cdot q^{j \cdot 0 \cdot j} \cdot a^{j \cdot 0} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \begin{bmatrix} j \\ j \end{bmatrix}_q \cdot (q^2;q^2)^{-0} \cdot$ 

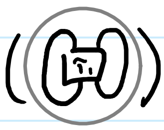
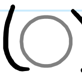
$P($  $) := \sum_{j \geq 0}$ 

$= \sum_{j \geq h \geq 0} (-q)^{h-j}$ 

$= \sum_{(h, j-h) \in \mathbb{N}^{1+1}} (-q)^{\ } \cdot q^{0} \cdot a^{0} \cdot \begin{bmatrix} h \\ h \end{bmatrix}_q \begin{bmatrix} j-h \\ j-h \end{bmatrix}_q \cdot (q^2;q^2)^{-0} \cdot$ 

3&4) <u>Idea</u>: if $P(T_1)$ and $P(T_2)$ have quiver form, then compute

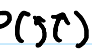
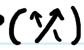
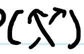
$P($  $)$ via e.g.  $\in \mathrm{span}_A \Big($  $\Big| \, 0 \leq k \leq j \Big)$

and $P($  $)$ via e.g.  $\in \mathrm{span}_A ($  $) \cong A$

Q: are these again in quiver form?

A: not quite, but almost!

1') $\exists$ refined notion of quiver form for $P(T)$,
it depends on connectivity & orientations
it involves extra q-Pochhammer symbols, fewer s.i.

2') $P($  $)$, $P($  $)$, $P($  $)$ satisfy this

2') $P(\;)$, $P(\;)$, $P(\;)$ satisfy this

3 & 4) A = yes $\blacksquare$

Remarks

- quivers for knots are ~~not~~ unique
  but for rational links, we have distinguished ones in SW17
- they have p vertices for $p = \det(K_{p/q})$
- $\exists$ geometric interpretation for knots-quivers correspondence (KQC) (EKL) but not for tangles!

Some Q :

- KQC beyond 4-ended tangles?
- how special are mDT series among q-hol. series?
- incorporating other colors?
- useful for conj. on growth?
- relation coHa $\leftrightarrow$ knot homology?