Higher Rank Rational (q-t)-Catalan Polynomials and a Finite Shuffle Theorem

Nicolle González (UC Berkeley)

Classical Story

$$\mathrm{Sym} = \mathbb{Q}(q,t)[x_1, x_2, \dots]^{S_\infty}$$

$$\mathrm{Sym}(r) = \mathbb{Q}(q,t)[x_1, x_2 \dots x_r]^{S_r}$$

The space of Diagonal Harmonics is:

$$DH_n = \left\{ f \in \mathbb{C}[x_1 \dots x_n, y_1 \dots y_n] \;\middle|\; \sum_{i=1}^{n} \frac{\partial^a}{\partial x_i} \frac{\partial^b}{\partial y_i} (f) = 0 \;\; \forall a+b>0 \right\}$$

It is an S_n-module under the diagonal action.

Haiman: $\mathrm{Frob}_{q,t}(DH_n) = \nabla e_n$

e_n: nth elementary sym. funct.

∇ = ubiquitous operator on Sym w/ eigenfunctions Macdonald polyn.

Shuffle Theorem: (Conj by Haglund-Haiman-Loehr-Remmel-Ulyanov)
(Proven by Carlsson-Mellit 14 years later)

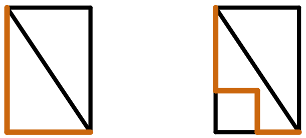
$$\nabla e_n = \sum_{\pi \in PF_n} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} x_{\mathrm{pides}(\pi)}$$

Sym. in x's
Sym in q,t.

Set $\gcd(m,n)=1$

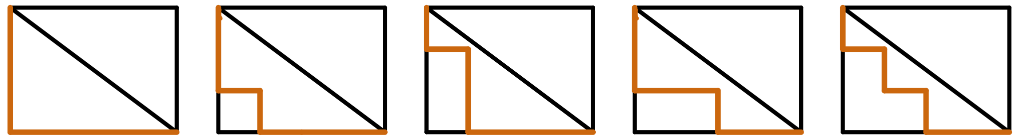
(m,n)-Dyck path: lattice path in $n \times m$ grid below diagonal, starts @ $(n,0)$ ends @ $(0,m)$.

$m=3$
$n=2$



coarea $=0$ area $=1$ | coarea $=1$ area $=0$

$m=3$
$n=4$



coarea $=0$, area $=3$ | 1, 2 | 2, 1 | 2, 1 | 3, 0

area = # boxes between path & diagonal

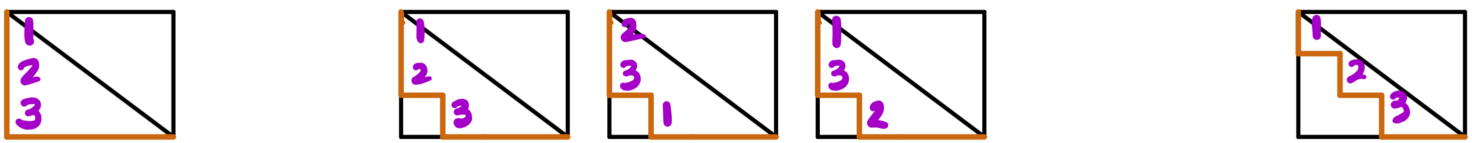
coarea $= \frac{(m-1)(n-1)}{2} -$ area $=$ # boxes in partition traced by the path

(m,n)-Dyck paths = Rational (q,t)-Catalan number

$$C_{(m,n)} := \frac{1}{n}\binom{n+m-1}{m}$$

(m,n)-Parking Function: $\pi = (D, \varphi)$ s.t. $D = (m,n)$-Dyck path
φ - bijection from North steps $D \to \{1,\dots,m\}$ that is decreasing on vertical runs.

Ex:



$\times 6$

Shuffle Theorem: $$\nabla e_n = \sum_{\underbrace{\pi \in PF_n}_{(n,n+1)\text{-}PF.}} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} x_{\mathrm{pides}(\pi)}$$

Rational Shuffle Theorem

While trying to compute the Khovanov-Rozansky homology of (m,n) torus knots, Gorsky-Negut conjectured a generalization of the shuffle theorem to the (m,n)-case.

The Elliptic Hall algebra $\mathcal{E}$ is the Hall algebra of the cat. of coherent sheaves on an elliptic curve.

$\Rightarrow$ $\mathcal{E}^{++}$ = $\mathbb{C}(q,t)$-algebra gen. by $\{ P_{(m,n)} \mid m,n \in \mathbb{Z}_{\geq 0} \}$ mod <u>many</u> relations.

Shiffmann-Vasserot: $\mathcal{E}^{++}$ acts on Sym (explicit geom. act.)

Rational Shuffle Thm. (proven by Mellit)

$$P_{(m,n)}(1) = \underbrace{\sum_{\pi \in PF_{(m,n)}} q^{\text{area}(\pi)} t^{\text{dinv}(\pi)} Q_{Des(\sigma^{-1})}(x)}_{:= \mathcal{H}_{(m,n)}(X;q,t) \text{ Hikita Polynomial.}}$$

} q,t sym. & X sym Funct.

where:

$S \subseteq \{1,\dots,m\} \Rightarrow Q_S(x) = \sum_{\substack{i_1 \leq \dots \leq i_m \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_m}$ Gessel's Q.S. functions "interpolate" between e's & h's.

For $\sigma \in \tilde{S}_m \Rightarrow Des(\sigma^{-1}) = \{1 \leq j \leq m \mid \sigma^{-1}(j+1) < \sigma^{-1}(j)\}$ Descent set of σ^{-1}

(Note: this is OK since $\exists$ map $PF_{m,n} \to \tilde{S}_m$ explained soon!)

AND if $L_{\frac{m}{n}}$ - unique f.d. irred. rep. of rational Cherednik alg. @ $\frac{m}{n}$

Hikita: Frob. charact. of (a filtration of) $L_{\frac{m}{n}}$ is $\mathcal{H}_{(m,n)}(X;q,t)$.

Question: is there a "finite" version of this over $Sym(r)$?

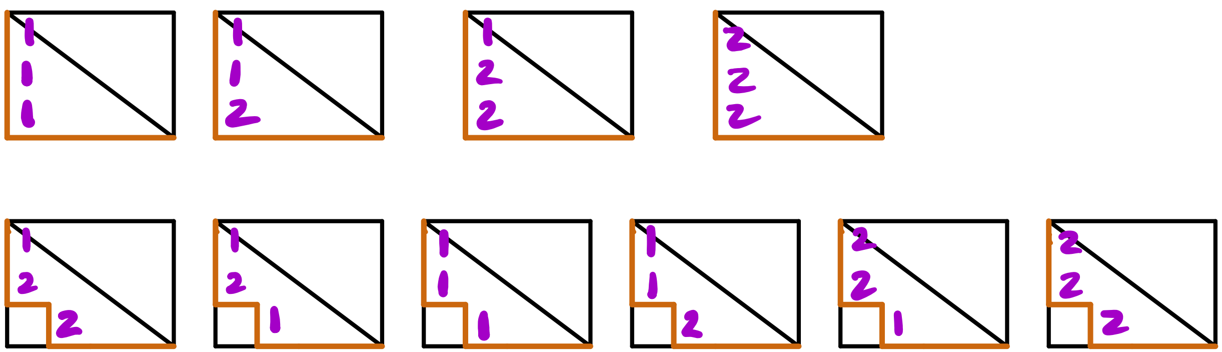
① Rank r Semi standard (m,n)-Parking Functions: $\pi = (D, \varphi)$

where $D = (m,n)$-Dyck path

φ: northsteps$(\pi) \longrightarrow \{1, \ldots, r\}$ that is weakly decreasing on vertical runs

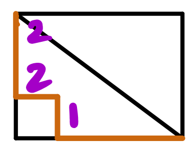
Ex: $r = m \to$ usual (m,n)-PF

$m = 3$
$n = 4$
$r = 2$



area(π) - same as before

$\mathrm{wt}(\pi) = (|\varphi^{-1}(1)|, |\varphi^{-1}(2)|, \ldots, |\varphi^{-1}(r)|)$

$\mathrm{wt}\Bigg($  $\Bigg) = (1,2)$

$r = m \to \pi \in PF_{(m,n)}$ $\mathrm{wt}(\pi) = (1, 1, \ldots, 1) = (1^m)$

② $SSPF^r_{(m,n)}$ are affine compositions.

Recall: $\widehat{S}_m = \{ \text{bij } \sigma : \mathbb{Z} \to \mathbb{Z} \mid \sigma(x+m) = \sigma(x) + m \text{ and } \sum_{i=1}^{m} \sigma(i) = \binom{m+1}{2} \}$

$\sigma \in \widehat{S}_m$ is n-stable if $\sigma(x+n) \geq \sigma(x)$. $\widehat{S}^n_m$ = n-stable affine perm.

Gorsky-Mazin-Vazirani: Gave a bijection $PF_{(m,n)} \longrightarrow \widehat{S}^n_m$

An (m,r)-affine composition is a function $f: \mathbb{Z} \to \mathbb{Z}$ s.t.:

① $f(x+m) = f(x) + r$

② $f^{-1}\{1, \dots, r\}$ has exactly one element for each res. class mod m

③ $\displaystyle\sum_{x \in f^{-1}\{1 \dots r\}} x = \binom{m+1}{2}$

Ex:

$m=5$
$r=3$
$n=4$

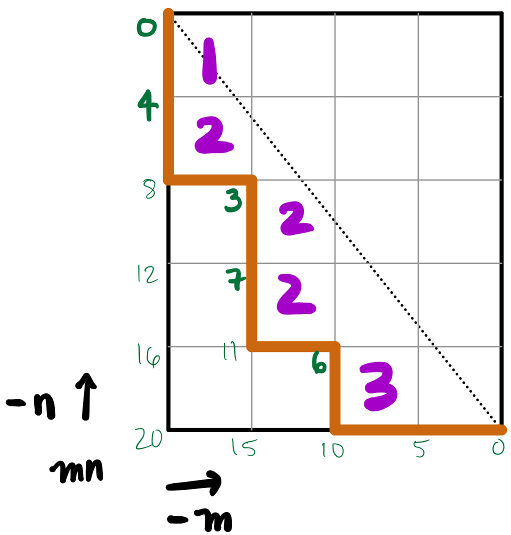
...	-1	0	1	2	3	4	5	6	7	8	9	10	...
	-1	1	0	-1	2	2	4	3	2	5	5	7	

+3 +3

Recipe:

$m=5$
$n=4$
$r=3$

$\pi \in SSPF^{3}_{(5,4)}$



…	−1	0	1	2	3	4	5	6	7	8	9	10	…
…	−1	1	0	−1	2	2	4	3	2	5	5	7	…

$\Downarrow$ shift domain by $k = \dfrac{n(m-1)-(m+1)}{2} - \sum_{j=1}^{m} a_j$

a_j = x-coordinate of jth north step.

$(k=+1)$

…	−2	−1	0	1	2	3	4	5	6	7	8	9	…
…	−1	1	0	−1	2	2	4	3	2	5	5	7	…

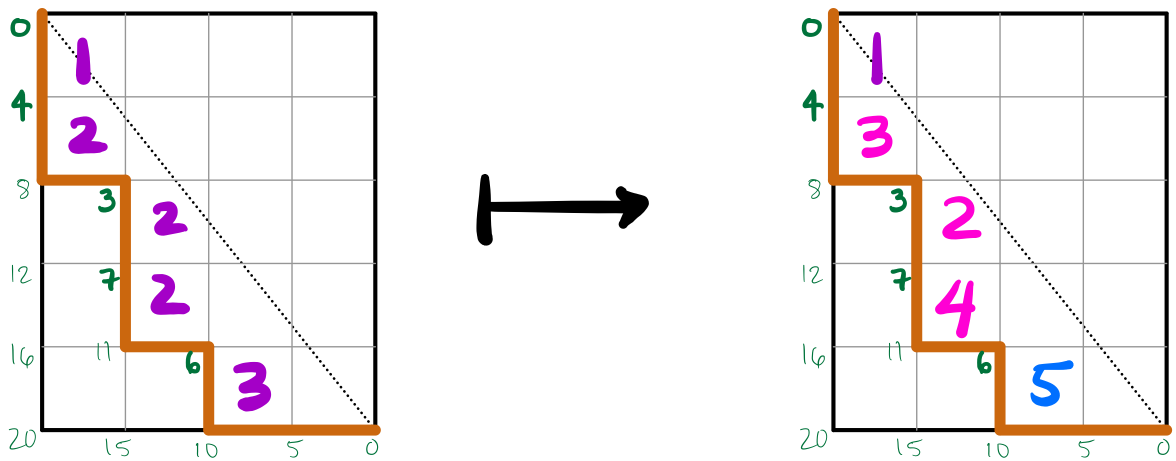
Thm (G-Simental-Vazirani) This map is a wt preserving bijection

$\mathcal{A}: SSPF^{r}_{(m,n)} \longrightarrow$ n-stable (m,r)-affine compositions.

@ $r=m \Rightarrow$ Gorsky-Mazin-Vazirani construction.

Next goal: Define statistics.

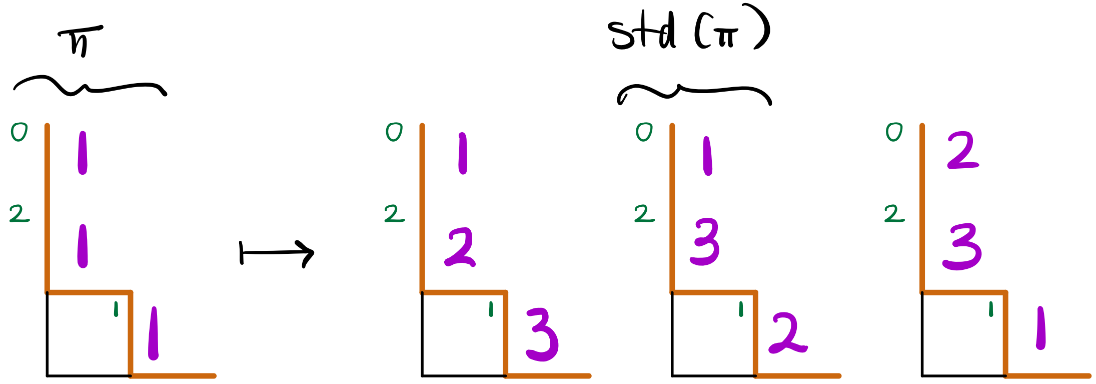
③ Standardization: To any $SSPF^{r}_{(m,n)}$ you can assign a "standardized" $PF_{(m,n)}$ as follows:



break ties using the Anderson labels

Ex:

$m = 3$
$n = 2$



[1,3,2] [1,2,3] [2,1,3]

↑ no inversions!

Given $\alpha \in PF_{(m,n)}$ w/ $\mathcal{A}(\alpha) = \sigma \in \widetilde{S}_m^n$, define:

$$\mathrm{codinv}(\alpha) := \left| \left\{ (i,h) \;\; \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix} \;\; \text{s.t. } \sigma(i+h) < \sigma(i) \right\} \right|$$

w/ $\mathrm{dinv}(\alpha) = \frac{(m-1)(n-1)}{2} - \mathrm{codinv}(\pi)$.

So for any $\pi \in SSPF_{(m,n)}$, set $\mathrm{dinv}(\pi) = \mathrm{dinv}(\mathrm{std}(\pi))$.

Comment: all of this is geometrically motivated: (see Etingof-Krylov-Losev-Simental)

$SSPF^r_{(m,n)}$ – index affine spaces in an affine paving of a parabolic affine Springer fiber

codinv — dimension of these affine spaces

The rank r rational (q,t)-Catalan polynomial is:

$$C^{(r)}_{(m,n)}(x_1, \ldots, x_r ; q,t) := \sum_{\pi \in SSPF^r_{(m,n)}} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} x^{\mathrm{wt}(\pi)}$$

Ex: at $r=1$ $\quad C^{(1)}_{(m,n)} = x_1^m \underbrace{C_{(m,n)}}_{\text{(q,t)-rational catalan number}}$

Recall: $$\mathcal{H}_{(m,n)}(x;q,t) = \sum_{\pi \in PF_{(m,n)}} q^{\text{area}(\pi)} t^{\text{dinv}(\pi)} \underbrace{Q_{Des(\sigma^{-1})}(x)}_{\text{Gessel Q.S. funct.}}$$

$$Q_S(x) = \sum_{\substack{i_1 \leq \ldots \leq i_m \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \ldots x_{i_m}$$

Prop: $$Q_{Des(\sigma^{-1})}(x)\Big|_{\substack{x_{r+i}=0 \\ i>0}} = \sum_{\pi \in std^{-1}(\alpha)} x^{wt(\pi)} \quad \text{w/} \quad \begin{array}{l} \alpha \in PF_{m,n} \\ \sigma = \mathcal{A}(\pi) \in \tilde{S}_m^n \end{array}$$

Thm (G-Simental-Vazirani)

$$\mathcal{H}_{(m,n)}(x;q,t)\Big|_{\substack{x_{r+i}=0 \\ i>0}} = C^{(r)}_{(m,n)}(x_1,\ldots,x_r;q,t)$$

Thus $C^{(r)}_{(m,n)}$ is q,t sym, x-sym, & schur positive.

Spherical DAHA and Finite Shuffle Thm

<u>Recall:</u> $\mathcal{E}^{++} = \mathbb{C}(q,t)$-alg gen by $\{ P_{(m,n)} \mid m,n \in \mathbb{Z}_{\geq 0} \}$

<u>Shiffman-Vasserot</u> $\mathcal{E}^{++} \curvearrowright$ geom action $(\mathbb{C}^\times)^2$- equivariant K-theory of Hilbert schemes of pts on $\mathbb{C}^2$ } complicated to explain

made very explicit by Negut. this K-theory $\simeq$ Sym.

Fixed point of this action yield Macdonald polynomials $\tilde{H}_\lambda$

<u>Rational Shuffle Thm:</u> $\underbrace{P_{m,n}(1)} = \mathcal{H}_{(m,n)}(X; q,t)$

$1 \in$ Sym under geom. action of $\mathcal{E}^{++}$.

<u>Spherical DAHA:</u> The double affine Hecke alg (of type $\mathfrak{gl}_r$) $\mathbb{H}(r)$ is the $\mathbb{C}(q,t)$-alg. gen by $T_i, X_j^{\pm}, Y_j^{\pm}$ w/ $1 \leq i < r$, $1 \leq j \leq r$ mod relations.

The spherical DAHA $\mathbb{SH}(r)$ is the spherical subalg. of $\mathbb{H}(r)$.

$$\mathbb{SH}(r) = e_{triv}\, \mathbb{H}(r)\, e_{triv}$$

$\mathbb{SH}(r)$ has generators $\{ P^{(r)}_{(m,n)} \mid m,n \in \mathbb{Z} \}$.

Action on $\mathrm{Sym}(r)$ $\exists$ a classical faithful polyn. representation of $\mathbb{SH}(r)$ on $\mathrm{Sym}(r) = \mathbb{Q}(q,t)[x_1 \ldots x_r]^{S_r}$

where T_i — Demazure Lusztig ops, X_i — mult, Y_i — Cherednik op.

There exists a natural map
$$\mathbb{SH}(r) \longrightarrow \mathbb{SH}(r-1)$$
$$P^{(r)}_{(m,n)} \longmapsto P^{(r-1)}_{(m,n)}$$

Schiffmann-Vasserot: $\mathcal{E}$ arises as an inverse limit of $\mathbb{SH}(r)$

$$\mathcal{E}^{++} = \varprojlim_{r} \mathbb{SH}(r)^{++}$$

$$P_{(m,n)} \longleftrightarrow P^{(r)}_{(m,n)}$$

Also $\mathrm{Sym} = \varprojlim_{r} \mathrm{Sym}(r)$

Hence, the polynomial action of $\mathbb{SH}(r)^{++}$ on $\mathrm{Sym}(r)$ induces a "polyn." action of $\mathcal{E}^{++}$ on $\mathrm{Sym}(r)$

$$
\begin{tikzcd}
\mathcal{E}^{++} \arrow[d, "\text{geom}"'] & & \mathcal{E}^{++} \arrow[d, "\text{polyn.}"'] & \mathbb{SH}(r)^{++} \arrow[l, "\lim"'] \arrow[d, "\text{polyn.}"] \\
\mathrm{Sym} \arrow[rr, phantom, "\overset{??}{\longleftrightarrow}"] & & \mathrm{Sym} & \mathrm{Sym}(r) \arrow[l, "\lim"]
\end{tikzcd}
$$

Thm (G-Simental-Vazirani) The polyn action and geom action of $\mathcal{E}^{++}$ on $\mathrm{Sym}(r)$ are isomorphic via the map sending

$$F[X; q, t] \longmapsto F\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$

Note: This is the exact same map that exchanges mac. polyn. $P_\lambda(X; q, t) \longmapsto \tilde{H}_\lambda(X; q, t)$ mod. mac.

Hence $\exists$ a "geom" action of $\mathbb{SH}(r)$ on $\mathrm{Sym}(r)$ that is nontrivially isomorphic to the polyn. action s.t.

$$\begin{tikzcd} \mathcal{E}^{++} \arrow[d, "\text{geom}"'] & \mathbb{SH}(r) \arrow[l, "\lim"'] \arrow[d, "\text{geom.}"] \\ \mathrm{Sym} & \mathrm{Sym}(r) \arrow[l, "\lim"] \end{tikzcd}$$

Thm (G.-Simental-Vazirani) $\exists$ an action of $\mathbb{SH}(r)$ on $\mathrm{Sym}(r)$ s.t.

$$P^{(r)}_{(m,n)}(1) = C^{(r)}_{(m,n)}(x_1, \ldots, x_r; q, t)$$

and which is nontrivially iso to the polyn. rep'n of $\mathbb{SH}(r)$.

Question: Can we describe this action explicitly?

Complicated even at $r=1$. $\mathbb{SH}(1)$ is simple but $C_{(m,n)}(q,t)$ is <u>not</u>.
Plethysm doesn't really work for finite # variables, etc....

Arigatou!