Plethysms, polynomial representations of linear groups and Hermite reciprocity over an arbitrary field

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Outline

§1 Motivation: the Wronskian isomorphism

§2 Plethysm and polynomial representations of $\mathrm{GL}_d(\mathbb{C})$

§3 Plethysms for $\mathrm{SL}_2(\mathbb{C})$ and Stanley's Hook Content Formula

§4 Modular plethystic isomorphisms

Sections 2 and 3 are with **Rowena Paget**, based on

- *Plethysms of symmetric functions and representations of* $\mathrm{SL}_2(\mathbb{C})$, arXiv:1907.07616, July 2019
  To appear in Journal of Algebraic Combinatorics.

Sections 1 and 4 are with my Ph.D student **Eoghan McDowell**, based on

- *Modular plethystic isomorphisms for two-dimensional linear groups*
  arXiv: by this Friday

§1 Motivation: A modular Wronskian isomorphism

Let V be a vector space.

- $$\begin{aligned} \mathrm{Sym}^2 V &= V^{\otimes 2}/\langle v \otimes w - w \otimes v : v, w \in V \rangle \\ &= \langle vw : v \in V, w \in V \rangle \end{aligned}$$
- $$\begin{aligned} \bigwedge^2 V &= V^{\otimes 2}/\langle v \otimes v : v \in V \rangle \\ &= \langle v \wedge w : v \in V, w \in V \rangle \end{aligned}$$

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Observation. $\mathrm{Sym}^2\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^{n+1}$ both have dimension $\binom{n+1}{2}$.

- For instance, if $v_1, \ldots, v_n$ is a basis for $\mathbb{C}^n$ then $\mathrm{Sym}^2\mathbb{C}^n$ has basis $v_1^2, \ldots, v_n^2, v_1v_2, \ldots, v_{n-1}v_n$ of size $n + \binom{n}{2}$.

Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

§1 Motivation: the Wronskian isomorphism

Are there nice isomorphisms $\mathrm{S}^2(k^n) \cong \Lambda^2(k^{n+1})$?

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This might be forced to migrate to math.SE but let me still risk it.

The spaces $\mathrm{S}^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ● 3 ● 50 ● 125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m \mathrm{S}^{m+r-1}(E) \cong \mathrm{S}^m\mathrm{S}^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $\mathrm{S}^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2\mathrm{S}^n(E) \to \mathrm{S}^2\mathrm{S}^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $\mathrm{S}^n(E) \cong k^{n+1}$ and $\mathrm{S}^{n-1}(E) \cong k^n$, so we have the required isomorphism $\mathrm{S}^2 k^n \cong \wedge^2 k^{n+1}$.

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ● 1 ● 32 ● 51

Action of $\mathrm{GL}_2(\mathbb{C})$ on $\langle X, Y\rangle$

$$\begin{array}{cc} X & Y \\ \end{array}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{array}{ccc} X^2\wedge XY & Y^2 \wedge XY & X^2 \wedge Y^2 \end{array}$$

$$\begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\gamma\beta^2 \\ \alpha\gamma^2\delta - \alpha\gamma^2\delta & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \gamma^2\alpha\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix}$$

$$\begin{array}{ccc} X^2\wedge XY & Y^2 \wedge XY & X^2 \wedge Y^2 \end{array}$$

$$= \begin{pmatrix} \alpha^2\Delta & -\beta^2\Delta & 2\alpha\beta\Delta \\ -\gamma^2\Delta & \delta^2\Delta & -2\gamma\delta\Delta \\ \alpha\gamma\Delta & -\beta\delta\Delta & (\alpha\delta+\beta\gamma)\Delta \end{pmatrix}$$

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$$= \begin{pmatrix} \alpha^2 & -\beta^2 & 2\alpha\beta \\ -\gamma^2 & \delta^2 & -2\gamma\delta \\ \alpha\gamma & -\beta\delta & (\alpha\delta+\beta\gamma) \end{pmatrix}$$

- This is not the matrix for $\mathrm{Sym}^2\mathbb{C}^2$.
- Instead it is (after a sign flip), the matrix for the dual
$$\mathrm{Sym}_2 E = \langle X\otimes X, Y\otimes Y, X\otimes Y + Y\otimes X\rangle.$$
- So what we've shown is that $\bigwedge^2 \mathrm{Sym}^2\mathbb{C}^2 \cong \mathrm{Sym}_2\mathbb{C}^2$.

Duality and the modular Wronskian isomorphism

Theorem (McDowell–W 2020)

Let F be any field. Let $E \cong F^2$ be the natural representation of $\mathrm{SL}_2(F)$. There is an isomorphism

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(F)} \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

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Outline of proof.

- Guess the right map. For instance, for $r = 2$ and $\ell = 3$, two cases are
  - $X^2Y \otimes XY^2 + XY^2 \otimes X^2Y \mapsto X^3Y \wedge XY^3 + X^2Y^2 \wedge X^2Y^2$
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- Prove it is injective. (Not obvious.)

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  - $X^2Y \otimes X^2Y \mapsto X^3Y \wedge X^2Y^2$.
- Prove it is injective. (Not obvious.)
- Prove is is $\mathrm{SL}_2(F)$-equivariant. (Highly not obvious.)

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- Polynomial representations of $\mathrm{GL}(E)$; take $E = \mathbb{C}^3$

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  - $E \otimes E \otimes E \cong \mathrm{Sym}^3 E \oplus \bigwedge^3 E \oplus \nabla^{(2,1)} E \oplus \nabla^{(2,1)} E$
  - Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$

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  - $s_{(2)}(x_1, x_2, x_3) \quad = x_1^2 + x_2^2 + x_3^3 + x_1 x_2 + x_1 x_3 + x_2 x_3$

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    $s_{(2,1)}(x_1, x_2, x_3) = x^{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3&\end{smallmatrix}}$

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  - Multiplication: $s_{(2)}(x_1, x_2, x_3)^2$

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    $$s_{(2,1)}(x_1, x_2, x_3) = x^{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}}$$
  - Multiplication: $s_{(2)}(x_1, x_2, x_3)^2$
  - Evaluate at monomials: $s_{(2)}(x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$

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  - Evaluate at monomials: $s_{(2)}(x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$
  - **Plethysm** (from Greek $\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$): $(s_\nu \circ s_\mu)(x_1, x_2, \ldots)$

§3 Plethysms for $\mathrm{SL}_2(\mathbb{C})$

Theorem

Let λ and μ be partitions and let $\ell, m \in \mathbb{N}$. The following are eqv:

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m E$;

(ii) $(s_\lambda \circ s_{(\ell)})(q, q^{-1}) = (s_\mu \circ s_{(m)})(q, q^{-1})$;

(iii) $s_\lambda(q^\ell, q^{\ell-2}, \ldots, q^{-\ell}) = s_\mu(q^m, q^{m-2}, \ldots, q^{-m})$;

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(iii) $s_\lambda(q^\ell, q^{\ell-2}, \ldots, q^{-\ell}) = s_\mu(q^m, q^{m-2}, \ldots, q^{-m})$;

(iv) $C(\lambda) + \ell + 1 / H(\lambda) = C(\mu) + m + 1 / H(\mu)$

where $/$ is difference of multisets (negative multiplicities okay) and

- $C(\lambda) = \{j - i : (i,j) \in [\lambda]\}$ *is the multiset of contents of λ;*
- $H(\lambda) = \{h_{(i,j)} : (i,j) \in [\lambda]\}$ *is the multiset of hook lengths of λ.*

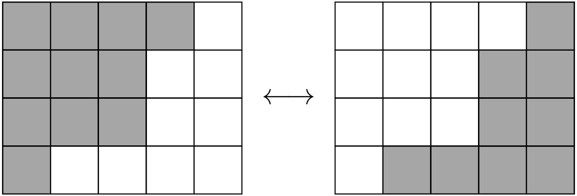
Part (iv) is a corollary of Stanley's Hook Content Formula.

Example. Part (iv) implies the Wronksian isomorphism (over $\mathbb{C}$).

Plethystic complement isomorphism for $\mathrm{SL}_2(\mathbb{C})$

Let λ be a partition contained in a box with d rows and s columns. Let $\lambda^{\bullet d}$ be its complement. For example if $s = 5$, $d = 4$ then

$$(4,3,3,1)^{\bullet 4} = (4,2,2,1).$$



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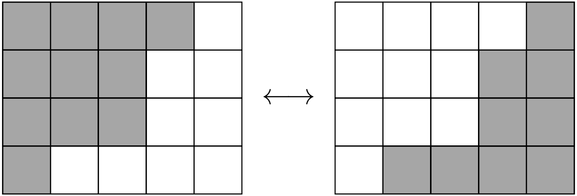


Theorem (King 1985 [if], Paget–W 2019 [only if])

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$. Let λ have at most d parts. Then

$$\nabla^\lambda \mathrm{Sym}^m E \cong \nabla^{\lambda \bullet d} \mathrm{Sym}^m E$$

if and only if $m = \ell = d - 1$.

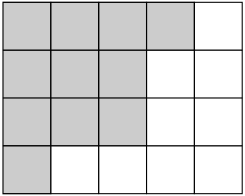
Stanley's HCF for the complement isomorphism

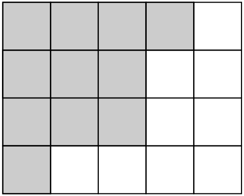
For example, using a rectangle with 4 rows and 5 columns,

$$\nabla^{(4,3,3,1)}\mathrm{Sym}^3 E \cong \nabla^{(4,2,2,1)}\mathrm{Sym}^3 E.$$

By Stanley's Hook Content Formula with $\lambda = (4,3,3,1)$, $\lambda^{\bullet 4} = (4,2,2,1)$

$$C(\lambda)+4\big/H(\lambda) = C(\lambda^{\bullet 4})+4\big/H(\lambda^{\bullet 4}).$$





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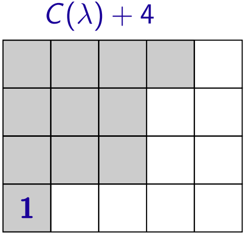
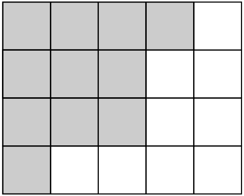
$$C(\lambda) + 4 \cup H(\lambda^{\bullet 4}) = C(\lambda^{\bullet 4}) \cup H(\lambda).$$





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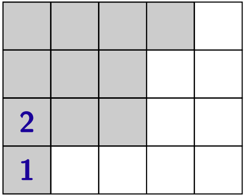
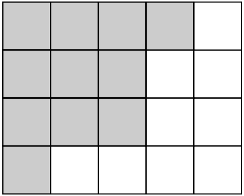
$C(\lambda) + 4$





Stanley's HCF for the complement isomorphism

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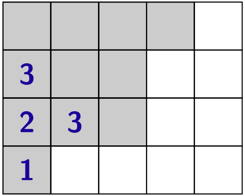

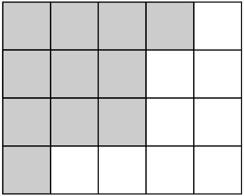

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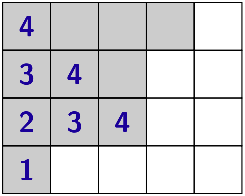

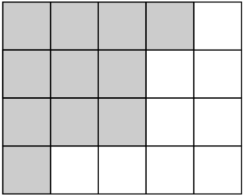

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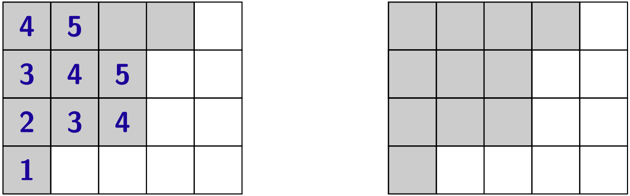

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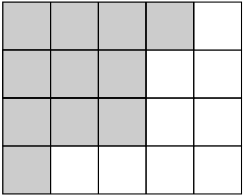
$C(\lambda) + 4$

4	**5**	**6**		
3	**4**	**5**		
2	**3**	**4**		
1				



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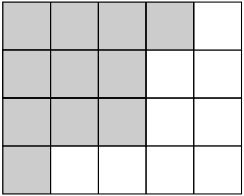
$C(\lambda) + 4$

4	**5**	**6**	**7**	
3	**4**	**5**		
2	**3**	**4**		
1				



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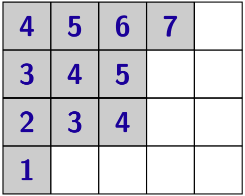
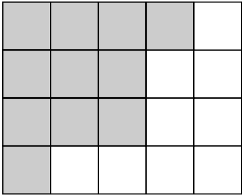


$H(\lambda)$



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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

			1	
		1		
1				

Stanley's HCF for the complement isomorphism

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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

			1	
		2		
	2	1		
1				

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4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

			1	
	3	2		
	2	1		
1				

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4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

		4	1	
	3	2		
4	2	1		
1				

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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

	5	4	1	
5	3	2		
4	2	1		
1				

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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

7	5	4	1	
5	3	2		
4	2	1		
1				

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$C(\lambda) + 4$

4	5	6	7	1
3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7	5	4	1	
5	3	2		
4	2	1		
1				

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4	5	6	7	1
3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7	5	4	1	1
5	3	2	3	2
4	2	1	4	3
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$C(\lambda^{\bullet 4}) + 4$

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4	5	6	7	1
3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^{\bullet 4})$

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7	5	4	1	1
5	3	2	3	2
4	2	1	4	3
1	7	6	5	4

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$C(\lambda) + 4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4_1
1_0	1_0	2_1	5_2	7_3

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
4_2	2_1	1_0	4_1	3_0
1_0	7_3	6_2	5_1	4_0

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Using a theorem of Bessenrodt: stronger version with arm lengths

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$C(\lambda) + 4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4_1
1_0	1_0	2_1	5_2	7_3

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
4_2	2_1	1_0	4_1	3_0
1_0	7_3	6_2	5_1	4_0

$C(\lambda^{\bullet 4}) + 4$

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Using a theorem of Bessenrodt: stronger version with arm lengths

Problem

Interpret this using Jack symmetric functions and prove a stronger symmetric functions identity

§4 Modular plethysms

Theorem (McDowell–W 2020)

- *Let G be a group;*
- *Let V be a d-dimensional representation of G over an arbitrary field;*
- *Let $s \in \mathbb{N}$, and let λ be a partition with $\ell(\lambda) \leq d$ and first part at most s.*
- *Recall that $\lambda^{\bullet d}$ denotes the complement of λ in the $d \times s$ rectangle.*

There is an isomorphism

$$\nabla^{\lambda} V \cong \nabla^{\lambda^{\bullet d}} V^{\star} \otimes (\det V)^{\otimes s}.$$

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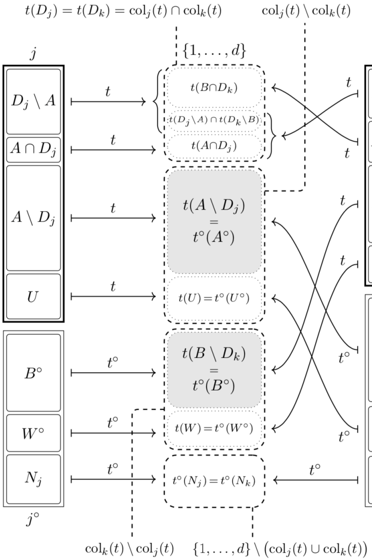
This generalizes the complementary partition isomorphism from $\mathrm{SL}_2(\mathbb{C})$ to arbitrary fields and groups.

One idea in proof: $\bigwedge^{\lambda'} V \cong \bigwedge^{(\lambda^{\bullet d})'} V$ up to determinants.

We show this isomorphism is compatible with the quotient map $\bigwedge^{\mu'} V \twoheadrightarrow \nabla^{\mu} V$ using generators and relations.



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This generalizes the complementary partition isomorphism to arbitrary fields and groups.

Corollary (Hermite 1854 over $\mathbb{C}$, McDowell–W 2020)

Let $m, \ell \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(F)$. Then $\mathrm{Sym}_m \mathrm{Sym}^{\ell} E \cong \mathrm{Sym}^{\ell} \mathrm{Sym}_m E$.

Obstructions to modular plethysms

Theorem (King 1985)

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$. For a large class of partitions λ, there is an isomorphism

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}(E)} \nabla^{\lambda'} \mathrm{Sym}^{\ell+\ell(\lambda')-\ell(\lambda)} E.$$

- In particular, King's result holds when λ is a hook; that is $\lambda = (a+1, 1^b)$ for some $a, b \in \mathbb{N}_0$.
- In Paget–W 2019 we showed that King's Theorem gives all plethystic isomorphisms relating $\nabla^\lambda \mathrm{Sym}^\ell E$ and $\nabla^{\lambda'} \mathrm{Sym}^m E$.
- King's result was (independently) reproved by Cagliero and Penazzi 2016.
- The special case of King's Theorem when λ is a rectangle is an instance of a theorem of Manivel 2007.

Obstruction to a modular generalization

Let F be an infinite field of prime characteristic p and let E be the natural representation of $\mathrm{SL}_2(F)$.

Theorem (McDowell–W 2020)

There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)}\mathrm{Sym}^{p^e+b}E$ by

- *Replacing ∇ with Δ (duality)*
- *Replacing $(a+1, 1^b)$ with $(b+1, 1^a)$ and $p^e + b$ with $p^e + a$ (King conjugation);*
- *Replacing $\mathrm{Sym}^\ell E$ with $\mathrm{Sym}_\ell E$ (another duality);*

are all non-isomorphic.

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Problem

What plethystic isomorphisms of representations of $\mathrm{SL}_2(\mathbb{C})$ have modular analogues?

Further work

Problem

What plethystic isomorphisms of representations of $\mathrm{SL}_2(\mathbb{C})$ *have modular analogues?*

Equivalences between two-row non-hook partitions: $a \geq b \geq 2$		
(c)	$(a,b)\,{}_{\ell}{\sim}_{\ell}\,(a,b)$	
(d)	$(a,a)\,{}_{c+1}{\sim}_{a+1}\,(c,c)$	(rectangular, Theorem 1.6), $c \geq 2$
(e)	$(a,b)\,{}_{2}{\sim}_{2}\,(a,a-b)$	(complement, Theorem 1.5), $a-b \geq 2$
(f)	$(2\ell,\ell+2)\,{}_{\ell}{\sim}_{\ell+2}\,(2\ell-2,\ell-2)$	$\ell \geq 4$

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(f)	$(2\ell,\ell+2)\,{}_{\ell}{\sim}_{\ell+2}\,(2\ell-2,\ell-2)$	$\ell \geq 4$

Problem

What other combinatorial identities have modular lifts?

For example, MacMahon's identity enumerating plane partitions in the $a \times b \times c$ box

$$\sum_{\pi \in \mathcal{PP}(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{q^{i+j+k-1}-1}{q^{i+j+k-2}-1}.$$

is equivalent to $\nabla^{(a^b)}\mathrm{Sym}^{b+c-1}E \cong_{\mathrm{SL}_2(\mathbb{C})} \nabla^{(b^a)}\mathrm{Sym}^{a+c-1}E$, and similar isomorphisms with all other permutations of a, b, c.