KLR-type presentation of affine Hecke algebras of type B

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OIST Representation Theory Seminar

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Introduction (Affine Hecke algebras)

Roughly speaking, affine Hecke algebras are associated to root systems, and they appear in:

- (originally) Reductive groups over a p-adic field: their representations control a large part of the representation theory of these reductive groups. (Iwahori–Matsumoto, Bernstein)

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- (for type A) Schur–Weyl duality with $U_q(\widehat{gl}_n)$. (Chari–Pressley)
- Invariants for links inside the solid torus (type A) and inside the double solid torus (type B/C).
- Integrable systems: Spin chains with one boundary (type A) or two boundaries (type B/C).

Introduction (of this talk)

For the study of affine Hecke algebras of type A (including also Ariki–Koike algebras, finite Hecke algebras of type A and B, the symmetric groups), we have the following tool:

KLR algebras of type A

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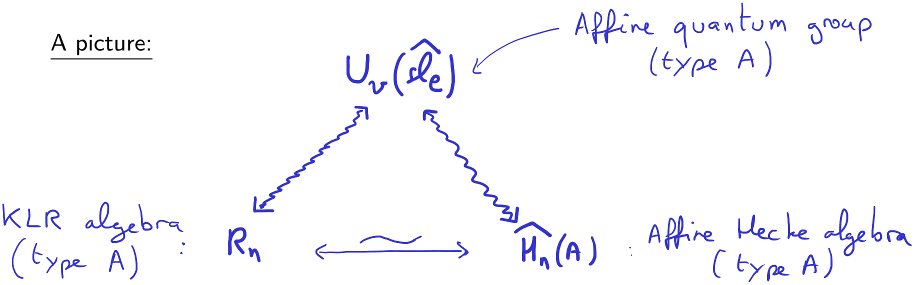
Outline of the talk:

1. Recall the type A situation.
2. Our main result (with R. Walker, '19): generalisation for types B/C,D.
3. Some applications

Type A

A picture:



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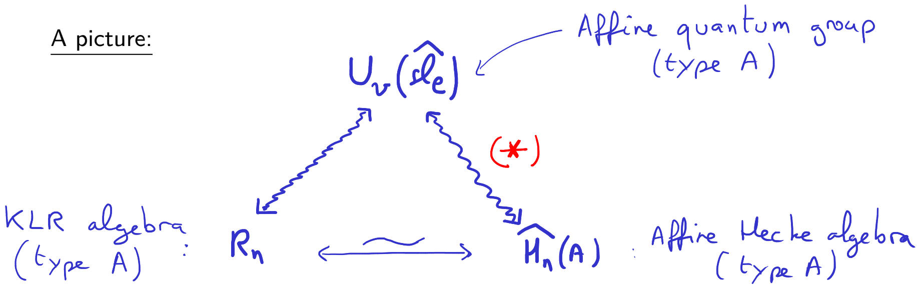

($\star$) Decomposition numbers of $\widehat{H}_n(A)$ can be calculated using the *canonical* or *crystal* basis of the affine quantum group $U_v(\widehat{sl}_e)$:

$$\text{dec. numbers} \quad = \quad P_{\lambda\mu}(v)|_{v=1} \ .$$

This was a conjecture of Lascoux–Leclerc–Thibon (LLT), generalised and proved by Ariki.

Type A

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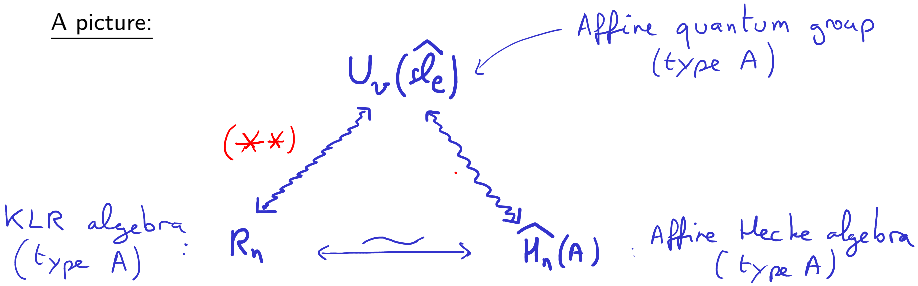

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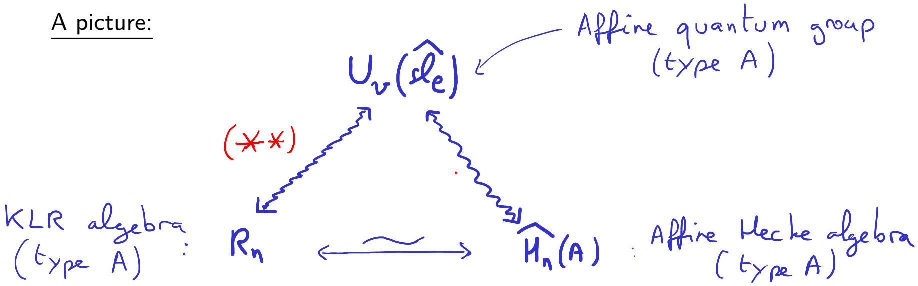

(⋆⋆) Categorification: R_n categorifies the negative half of $U_v(\widehat{sl}_e)$ and the cyclotomic quotients of R_n categorify the integrable highest-weight modules.

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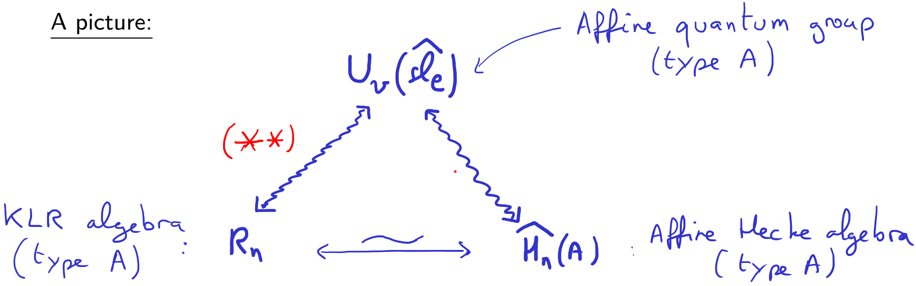

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Now R_n is $\mathbb{Z}$-graded. And its graded decomposition numbers can also be calculated using the canonical basis:

$$\boxed{\text{graded dec. numbers of KLR} \;=\; P_{\lambda\mu}(v)\ .}$$

(Khovanov–Lauda, Rouquier, Kang–Kashiwara, Brundan–Kleshchev)

Type A

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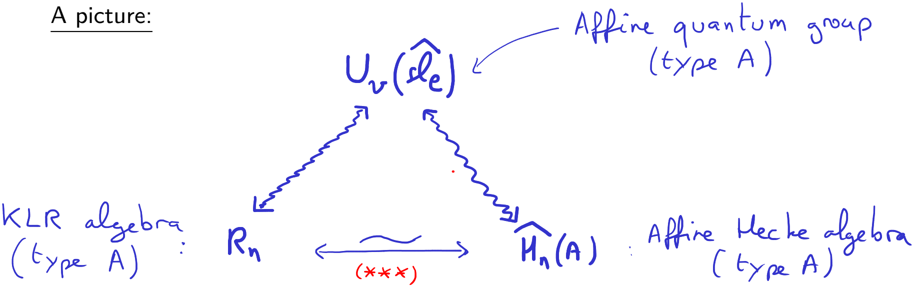

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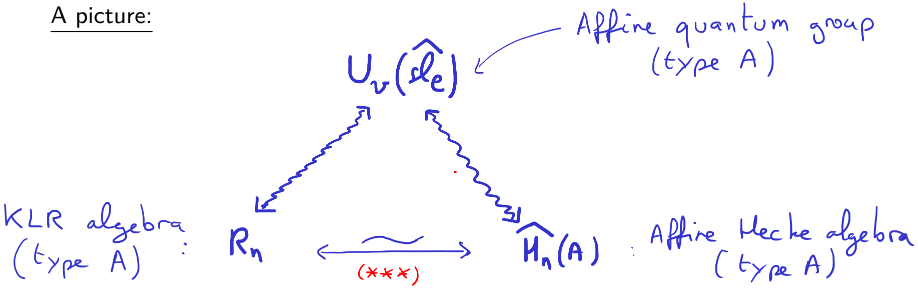

$(\star\star\star)$ Thm.: cycl. quotients of $R_n \cong$ cycl. quotients of $\widehat{H}_n(A)$

(Brundan–Kleshchev–Rouquier isomorphism)

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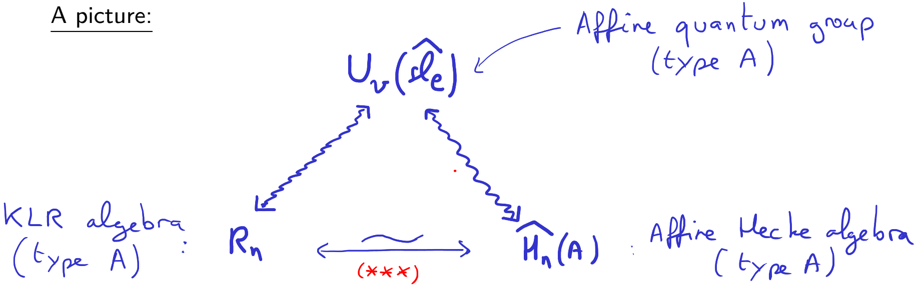

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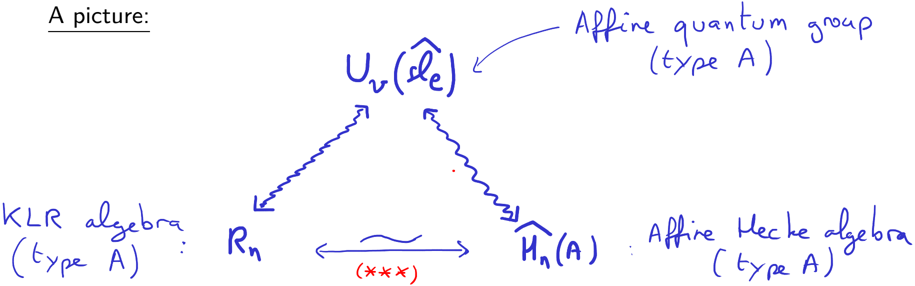

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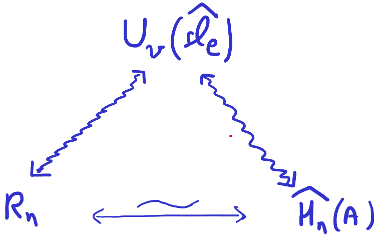
Rem.: Graded decomposition numbers contain more information than usual decomposition numbers, and are somehow easier.
Ex.: $P_{\lambda\mu}(v) = 1 + v + v^2$ instead of $P_{\lambda\mu}(1) = 3$.

Other types (B,C,D)



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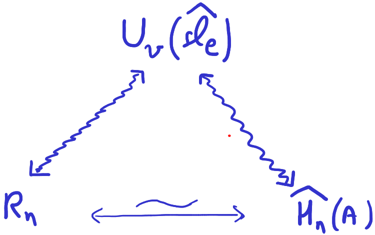
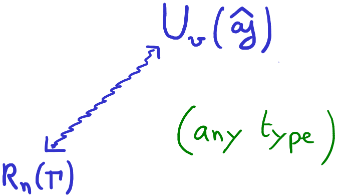


- Affine quantum groups side:



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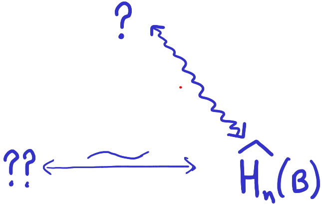

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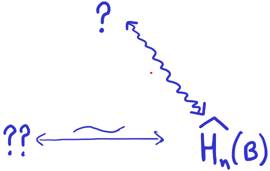

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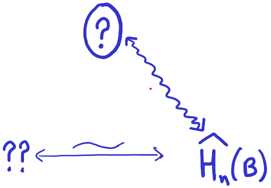

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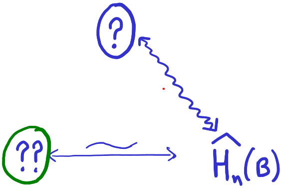

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In this special case $\rightsquigarrow$ an algebra which could maybe go into ??.

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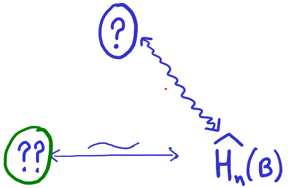

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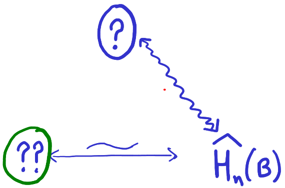

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Rem.:
– True for any cyclotomic quotients (no restriction on the representations)
– Same story for type D

More details (Brundan–Kleshchev isomorphism)

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$\widehat{H}_n(A)$: $g_1, \ldots, g_{n-1}, X_1, \ldots, X_n$ (one parameter: q) $(g_1 g_2 g_1 = g_2 g_1 g_2)$

- X_i's commute, g_i's satisfy braid relations and $g_i^2 = (q - q^{-1})g_i + 1$.
- Mixed relations: $g_i X_j - X_{s_i(j)} g_i = \ldots$ (Ex.: $g_i X_i = X_{i+1} g_i - (q - q^{-1}) X_{i+1}$)

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   $\rightsquigarrow$ idempotents $e(a_1, \ldots, a_n) =: e(\mathbf{a})$ (a_i eigenvalue of X_i)

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Summary: Commutative subalgebra $(X_1, \ldots, X_n)$ + Intertwining elements:

$\rightsquigarrow$ We need a clever way to renormalise the intertwining elements

Thm.: cycl. quotients of $\widehat{H}_n(A)$ $\cong$ cycl. quotients of R_n

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Miracle: We find an algebra R_n gen. by: $e(\mathbf{a})$, $y_1, \ldots, y_n$, $\psi_1, \ldots, \psi_{n-1}$.

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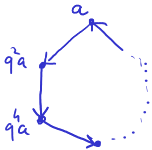
2. Parameter q (almost) disappeared from the relations: $\rightsquigarrow$ the algebra only depends on the quiver (*i.e.* on the order of q^2):



(multiplication by q^2) $\rightsquigarrow$ generalisation $R_n(\Gamma)$ for any quiver Γ.

Now, type B

$\widehat{H}_n(B): \quad g_1, \ldots, g_{n-1}, X_1, \ldots, X_n \ + \ g_0$ (two parameters: p, q)

$$\text{Rels of } \widehat{H}_n(A) \ + \quad \begin{aligned} & g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0 \,, \\ & g_0^2 = (p - p^{-1}) g_0 + 1 \,, \\ & g_0 X_1^{-1} = X_1 g_0 - (p - p^{-1}) X_1 \ . \end{aligned}$$

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$$\text{Rels of } \widehat{H}_n(A) \;+\; \begin{array}{l} g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0\,, \\ g_0^2 = (p - p^{-1}) g_0 + 1\,, \\ g_0 X_1^{-1} = X_1 g_0 - (p - p^{-1}) X_1\ . \end{array}$$

$\rightsquigarrow$ Still have: Commuting elements (X_i's) and intertwining elements.

$\rightsquigarrow$ Same procedure: Now renormalising G_0 into a new generator ψ_0.

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Rem.: Brundan–Kleshchev isomorphism for $\widehat{H}_n(A)$ does not extend as it is to $\widehat{H}_n(B)$.
We first had to modify it for type A.

Main result

We find an algebra V_n gen. by: $e(\mathbf{a})$, $y_1, \dots, y_n$, $\psi_1, \dots, \psi_{n-1}$ $+$ ψ_0.

Relations of KLR $+$

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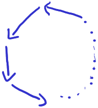
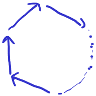
Theorem (LPA–Walker)

cycl. quotients of $\widehat{H}_n(B)$ $\cong$ cycl. quotients of V_n

$$\prod_a (X_1 - a)^{m_a} = 0 \quad \leftrightarrow \quad y_1^{m_{a_1}} e(\mathbf{a}) = 0 \ \ (\forall \mathbf{a}) \qquad \text{(for any choice of } m_a\text{'s)}$$

Applications

What we really need to define an algebra like V_n is:

1. A quiver Γ (formalising the relation $j = q^2 i \rightsquigarrow i \to j$):
   Ex.:   (does not have to be connected)

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   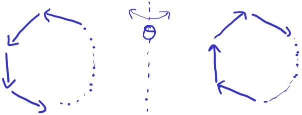

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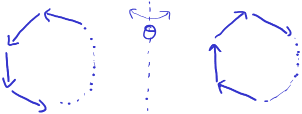
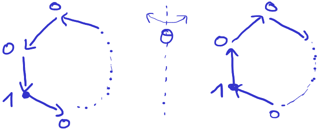
   

3. A framing vector $\vec{\lambda}$ (formalising the presence of special points $\{\pm p^{\pm 1}\}$):
   Ex.:
   

   ($\vec{\lambda} = 1$ on $p, -p, p^{-1}, -p^{-1}$)

(with S. Rostam) A class of algebras $V_n(\Gamma, \theta, \vec{\lambda}) \supset$ all V_n for $\widehat{H}_n(B)$.

$\rightsquigarrow$ All are graded with a faithful polynomial representation and a PBW basis.

Schur–Weyl duality

- Affine quantum group $U_q(\widehat{sl}_N) \leftrightarrow$ affine Hecke algebra (type A)

(Chari, Pressley, '95)

- Affine quantum groups (any type) $\leftrightarrow$ KLR algebras (any type)

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$\wr\updownarrow$ (type B: our isomorphism)

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(Algebras $V_n(\Gamma, \theta, \vec{\lambda})$ are called sometimes orientifold KLR algebras)

Dipper–Mathas theorem

- Arbitrary cycl. quotient of $\widehat{H}_n(A)$:

$$H_n(A)^{\Lambda} \ : \quad (X_1 - a)^{m_0}(X_1 - aq^2)^{m_1} \ldots\ldots \times (X_1 - b)^{m'_0}(X_1 - bq^2)^{m'_1} \ldots\ldots = 0$$

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- q^2-connected cycl. quotients: $H_n(A)^{\Lambda_1} \;:\; (X_1 - a)^{m_0}(X_1 - aq^2)^{m_1} \ldots = 0$

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Theorem (Dipper–Mathas '02)

$$H_n(A)^\Lambda \overset{Morita}{\sim} \bigoplus_{n_1+n_2=n} H_{n_1}(A)^{\Lambda_1} \otimes H_{n_2}(A)^{\Lambda_2}$$

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Proof. Actually (this can be seen "easily" on the KLR algebra side)

$$H_n(A)^\Lambda \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{\binom{n}{n_1}}(H_{n_1}(A)^{\Lambda_1} \otimes H_{n_2}(A)^{\Lambda_2})$$

Dipper–Mathas theorem for type B

$$H_n(B)^\Lambda \ : \quad \overbrace{\ldots(X_1 - a^{\pm 1}q^{2i})^{m_i}\ldots}^{\text{eig. in } a^{\pm 1}q^{2\mathbb{Z}}} \times \overbrace{\ldots(X_1 - b^{\pm 1}q^{2j})^{m_j'}\ldots}^{\text{eig. in } b^{\pm 1}q^{2\mathbb{Z}}} = 0$$

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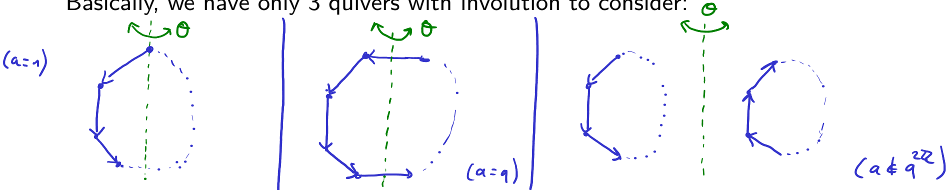
Proof. Same idea using $V_n(\Gamma)$ and a "disjoint quiver isomorphism"

Basically, we have only 3 quivers with involution to consider:



Perspectives

Graded representation theory of $\widehat{H}_n(B)$ (graded dec. numbers?)

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- One-boundary TL algebras (blob algebras): level 2 quotient of $\widehat{H}_n(A)$. ✓
- Two-boundary TL algebras (symplectic blob algebras): quotients of $\widehat{H}_n(B)$. ?

(*under investigation*)

Thank You