Integral Basic Algebras

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Motivation

(1) Let G be a finite group. Choose a suitable p-modular system $(K, \mathcal{O}, k)$ and suppose that $p \nmid |G|$. The representation theory of KG and kG are 'the same'. In particular, they are both semisimple and their simple modules are parametrised by the conjugacy classes of G.

(2) Fix a field $\mathbb{F}$ of characteristic p. There is a certain subalgebra (the descent algebra) of the group algebra $\mathbb{F}\mathfrak{S}_n$ which is usually not semisimple. The Ext quivers of the cases $p = \infty$ and $p \nmid |\mathfrak{S}_n|$ are identical and the algebras have the same representation type.

Notation

- $\mathbb{N}$ is the set consisting of non-negative integers
- $\mathbb{F}$ is an algebraically closed field with characteristic p (either $p < \infty$ or $p = \infty$)
- $[a, b]$ is the set consisting of integers n where $a \leq n \leq b$
- An $\mathbb{F}$-algebra A is assumed to be finite-dimensional unital associative algebra over $\mathbb{F}$ unless stated otherwise.

Basic Algebra

Definition 1

An $\mathbb{F}$-algebra A is basic if every simple A-module is one-dimensional.

- Any finite-dimensional algebra B is Morita equivalent to a unique (up to isomorphism) basic algebra A.

Path Algebra

Let Q be a (finite) quiver, that is, Q is a directed graph with the vertex set Q_0 and arrow set Q_1 are both finite. For an arrow $\alpha : v \to w \in Q_1$, we write $\mathsf{h}(\alpha) = v$ and $\mathsf{t}(\alpha) = w$. A path γ in Q is a finite sequence of arrows in Q of the form

$$\gamma = \alpha_\ell \cdots \alpha_2 \alpha_1$$

$$\gamma : \circ \xleftarrow[\alpha_\ell]{} \cdots \xleftarrow[\alpha_1]{} \circ$$

such that $\mathsf{t}(\alpha_i) = \mathsf{h}(\alpha_{i+1})$ for each $i \in [1, \ell - 1]$. In this case, $\mathsf{t}(\gamma) = \mathsf{t}(\alpha_\ell)$, $\mathsf{h}(\gamma) = \mathsf{h}(\alpha_1)$, and the length of γ is ℓ. The concatenation $\gamma\xi$ of two paths γ and ξ is defined if $\mathsf{t}(\xi) = \mathsf{h}(\gamma)$. For each vertex $i \in Q_0$, we write e_i for the path of length 0 at the vertex i.

Definition 2

Let Q be a quiver. The path algebra $\mathbb{F}Q$ is the vector space with a formal basis consisting of all paths in Q and, for any two paths γ and ξ, the multiplication is defined as

$$\gamma \cdot \xi = \begin{cases} \gamma\xi & \text{if } \mathsf{t}(\xi) = \mathsf{h}(\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathbb{F}Q$ is unital with the unit $\sum_{i \in Q_0} e_i$
- $\mathbb{F}Q$ is finite-dimensional if and only if Q has no oriented cycle
- $\mathbb{F}Q$ is basic
- For any $m \in \mathbb{N}$, the subspace of $\mathbb{F}Q$ spanned by paths of lengths at least m is an ideal of $\mathbb{F}Q$.

Example 1

Consider the quiver Q as follows:

Then $\mathbb{F}Q \cong \mathbb{F}[x]$.

The Ext groups

Let A be an $\mathbb{F}$-algebra, M, N be A-modules and

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

be the projective resolution of M. Take $\mathrm{Hom}_A(-, N)$, we get

$$C := 0 \to \mathrm{Hom}_A(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_A(P_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_A(P_2, N) \to \cdots .$$

The nth Ext group $\mathrm{Ext}_A^n(M, N)$ is the nth cohomology group of C. In particular, $\mathrm{Ext}_A^0(M, N) \cong \mathrm{Hom}_A(M, N)$.

The Ext Quiver

Let A be an $\mathbb{F}$-algebra and $\{S_i : i \in I\}$ be a complete set of non-isomorphic simple A-modules. We construct the Ext quiver Q_A of A as follows. Let Q_A be the quiver (directed graph) with vertices labelled by I. For $i, j \in I$, the number of arrows from i to j is

$$(\mathsf{Rad}(\mathsf{P}(S_i))/\mathsf{Rad}^2(\mathsf{P}(S_i)) : S_j) = \dim_{\mathbb{F}} \mathsf{Ext}^1_A(S_i, S_j)$$

where $\mathsf{P}(S_i)$ is the projective cover of S_i and $\mathsf{Rad}^n(V)$ is the nth radical of V (for any A-module V).

Theorem 3 (Gabriel 1979)

Let A be a basic $\mathbb{F}$-algebra, Q_A be the Ext quiver of A and J be the ideal of $\mathbb{F}Q_A$ consisting of all paths of lengths at least one. Then

$$A \cong \mathbb{F}Q_A/I$$

for some ideal I of $\mathbb{F}Q_A$ such that $J^n \subseteq I \subseteq J^2$ for some integer $n \geq 2$.

Example 2

If $p < \infty$, the Ext quiver Q of the group algebra $\mathbb{F}C_p$ is the single vertex with the single loop ε.

Then

$$\mathbb{F}C_p \cong \mathbb{F}Q/(\varepsilon^p) \cong \mathbb{F}[x]/(x^p).$$

Reduction Modulo p

- Fix a p-modular system $(K, \mathcal{O}, k)$, that is $\mathcal{O}$ is a local PID with the maximal ideal (π), K is the field of fractions of $\mathcal{O}$, and k is the residue field $\mathcal{O}/(\pi)$ of characteristic $p < \infty$.
- For an $\mathcal{O}$-torsion free $\mathcal{O}$-algebra A, let

$$\hat{A} = K \otimes_{\mathcal{O}} A, \qquad \bar{A} = k \otimes_{\mathcal{O}} A, \qquad J_A = A \cap \mathrm{Rad}(\hat{A}),$$

  and, for an $\mathcal{O}$-free A-module M, let

$$\hat{M} = K \otimes_{\mathcal{O}} M, \qquad \bar{M} = k \otimes_{\mathcal{O}} M$$

  be the $\hat{A}$- and $\bar{A}$-modules respectively.
- For simplicity, we may assume all $\mathcal{O}$-modules are finitely generated.

Integral Basic Algebra

Let $Q = (Q_0, Q_1)$ be a quiver. The quiver algebra $\mathcal{O}Q$ is the $\mathcal{O}$-free algebra with a formal basis consisting of all paths in Q and, for any two paths γ and ξ, the multiplication is defined as

$$\gamma \cdot \xi = \begin{cases} \gamma\xi & \text{if } \mathsf{t}(\xi) = \mathsf{h}(\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Let J_Q be the (two-sided) ideal of $\mathcal{O}Q$ consisting of paths of length at least one.

The Hypotheses (Z $\Rightarrow$ Y $\Rightarrow$ X $\Rightarrow$ W)

$$k = \mathcal{O}/(\pi) \quad \bar{A} = k \otimes_{\mathcal{O}} A \quad J_A = A \cap \mathrm{Rad}(\hat{A}) \quad \hat{S} = K \otimes_{\mathcal{O}} S$$

Hypothesis W

If M and S are finitely generated $\mathcal{O}$-free A-modules with $\hat{S}$ simple then $\mathrm{Ext}^t_A(M, S)$ is $\mathcal{O}$-free for all $t \geqslant 0$.

Hypothesis X

Suppose that the algebra A has finite rank over $\mathcal{O}$, we have $\mathrm{Rad}(A) = \pi A + J_A$ and orthogonal idempotent decompositions of the identity in $\bar{A}$ lift to A.

Hypothesis Y

There exist a finite quiver Q, an ideal I of $\mathcal{O}Q$, and $n \geqslant 2$ such that $J_Q^n \subseteq I \subseteq J_Q^2$ and $A \cong \mathcal{O}Q/I$ is $\mathcal{O}$-free of finite rank. (Equivalently, $\hat{A}$ is basic and A, A/J_A and A/J_A^2 are all $\mathcal{O}$-free.)

Hypothesis Z

Hypothesis Y holds, and for all $n \geqslant 1$, A/J_A^n is $\mathcal{O}$-free.

If A satisfies Hypothesis Y, there are $\mathcal{O}$-free of rank one A-modules $S_1, \ldots, S_m$ where $m = |Q_0|$ such that, for all $i = 1, \ldots, m$, $\hat{S}_i$ and $\bar{S}_i$ are simple $\hat{A}$- and $\bar{A}$-modules respectively. Let $\hat{P}_i$ and $\bar{P}_i$ be their projective covers.

Theorem 3 (Benson-L. 2024)

(i) *Hypothesis Z* $\Rightarrow$ *Hypothesis Y* $\Rightarrow$ *Hypothesis X* $\Rightarrow$ *Hypothesis W*

(ii) *If* A *satisfies Hypothesis W, then, for any finitely generated* $\mathcal{O}$*-free* A*-modules* M, S *with* $\hat{S}$ *simple and* $t \geqslant 0$*, we have* $\mathrm{Ext}^t_A(M, S)$ *is* $\mathcal{O}$*-free and*

$$k \otimes_{\mathcal{O}} \mathrm{Ext}^t_A(M, S) \cong \mathrm{Ext}^t_{\bar{A}}(\bar{M}, \bar{S}),$$
$$K \otimes_{\mathcal{O}} \mathrm{Ext}^t_A(M, S) \cong \mathrm{Ext}^t_{\hat{A}}(\hat{M}, \hat{S}).$$

In particular, $\dim_k \mathrm{Ext}^t_{\bar{A}}(\bar{M}, \bar{S}) = \dim_K \mathrm{Ext}^t_{\hat{A}}(\hat{M}, \hat{S})$.

(iii) *If* A *satisfies Hypothesis Z, then the radical layer multiplicities of* $\hat{P}_i$ *and* $\bar{P}_i$ *are equal, i.e., for all* $1 \leqslant i, j \leqslant m$ *and* $t \geqslant 0$,

$$(\mathrm{Rad}^t(\hat{P}_i)/\mathrm{Rad}^{t+1}(\hat{P}_i) : \hat{S}_j) = (\mathrm{Rad}^t(\bar{P}_i)/\mathrm{Rad}^{t+1}(\bar{P}_i) : \bar{S}_j).$$

Corollary 4

If A satisfies Hypothesis W, then the Ext quivers of $\hat{A}$ and $\bar{A}$ are identical.

Descent Algebras of Coxeter Groups

Let (W, S) be a Coxeter system. There is a length function $\ell : W \to \mathbb{N}$ defined as follows: for $w \in W$, let r be the smallest non-negative integer such that

$$w = s_1 \cdots s_r$$

where $s_1, \ldots, s_r \in S$. Define $\ell(w) = r$.

- $\ell(e) = 0$
- $\ell(s) = 1$ for any $s \in S$
- W has a unique longest element

For each subset $J \subseteq S$, let

$$W_J = \text{parabolic subgroup of } W \text{ generated by } J,$$
$$X_J = \text{distinguished left coset representatives consisting of minimal length elements for } W/W_J.$$

Let $\mathcal{O}$ be an integral domain and $\mathcal{O}W$ be the group algebra. Define

$$x_J = \sum_{w \in X_J} w \in \mathcal{O}W.$$

For subsets $J, K \subseteq S$, let X_{JK} be the distinguished double coset representatives of (W_J, W_K) in W.

Theorem 5 (Solomon 1976)

Let $\mathcal{O}$ be an integral domain. Let $\Gamma := \{x_J : J \subseteq S\}$ and $\mathscr{D}_{\mathcal{O}}(W) = \mathrm{span}_{\mathcal{O}}\Gamma \subseteq \mathcal{O}W$. Then $\mathscr{D}_{\mathcal{O}}(W)$ is a subalgebra of $\mathcal{O}W$ with $\mathcal{O}$-free basis Γ where

$$x_J x_K = \sum_{L \subseteq S} a_{JK}^{L} x_L$$

where a_{JK}^{L} is the number of elements $w \in X_{JK}$ such that $w^{-1}Jw \cap K = L$.

- The algebra $\mathscr{D}_{\mathcal{O}}(W)$ is known as the Solomon's descent algebra.
- It is usually neither commutative nor semisimple.

Theorem 6 (Solomon 1976, Atkinson-van Willigenburg 1997, Atkinson-Pfeiffer-van Willigenburg 2002)

The algebra $\hat{\mathscr{D}}$ *(respectively* $\bar{\mathscr{D}}$*) is basic.*

Theorem 7 (Benson-L. 2024)

The descent algebra $\mathscr{D} = \mathscr{D}_{\mathcal{O}}(W)$ *satisfies Hypothesis W when* $p \nmid |W|$*, and Hypothesis Z when* p *is sufficiently large. In particular, the Ext quivers of* $\hat{\mathscr{D}}$ *and* $\bar{\mathscr{D}}$ *are identical when* $p \nmid |W|$*.*

Example 8 (Type $\mathbb{A}$)

Let $\mathscr{D}_n = \mathscr{D}_{\mathcal{O}}(\mathbb{A}_{n-1})$ be the descent algebra of type $\mathbb{A}$. The simple modules for the descent algebras $\hat{\mathscr{D}}_n$ and $\bar{\mathscr{D}}_n$ are parametrised by partitions and p-regular partitions of n respectively. Their Ext quivers have previously been computed by Schocker when $p = \infty$ and Saliola when $p \nmid |\mathfrak{S}_n|$.

For example, when $n = 5$ and $p \neq 2, 5$, the Ext quiver of $\hat{\mathscr{D}}_5$ or $\bar{\mathscr{D}}_5$ is

$$21^3 \longrightarrow 31^2 \longrightarrow 41 \longrightarrow 5 \longleftarrow 32 \longleftarrow 2^2 1 \qquad 1^5$$

However, when $p = 5$, the Ext quiver of $\bar{\mathscr{D}}_5$ is

$$21^3 \longrightarrow 31^2 \longrightarrow 41 \longrightarrow 5 \circlearrowleft \longleftarrow 32 \longleftarrow 2^2 1$$

Other examples

Example 9

Let p be an arbitrary prime.

1. (i) The nil-Coxeter algebra of a finite Coxeter group satisfies Hypothesis Z.
2. (ii) The face algebra of hyperplane arrangements in a real space satisfies Hypothesis Y.
3. (iii) The 0-Hecke algebra of a finite Coxeter group satisfies Hypothesis X.

Trichotomy Theorem

Based on the isomorphism class of its indecomposable modules, a finite-dimensional algebra is classified into the following three types: representation finite type, tame type and wild type.

Theorem 4 (Drozd 1980)

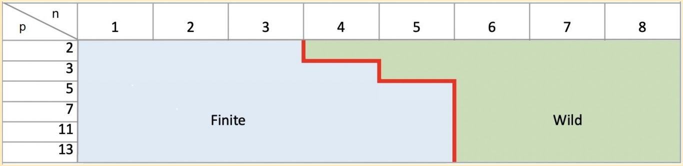
An algebra has either representation finite, tame or wild type and these three types are mutually exclusive.

Theorem 10 (Schocker 2004)

The descent algebra $\hat{\mathscr{D}}_n$ *has finite representation type if* $n \leqslant 5$, *and wild type otherwise.*

Theorem 11 (Erdmann-L. 2024)

The representation type $\bar{\mathscr{D}}_n$ *is depicted as follows:*



- Schocker's result can be seen as the asymptotic behaviour of our result when $p \to \infty$.

Problem

Suppose that A satisfies Hypothesis Z. Are the representation type of $\hat{A}$ and $\bar{A}$ the same, that is, $\hat{A}$ has finite type (respectively, tame or wild) if and only if $\bar{A}$ has finite type (respectively, tame or wild)?

References I

M. D. Atkinson, G. Pfeiffer, and S. J. van Willigenburg, *The p-modular descent algebras*, Algebras and Representation Theory **5** (2002), no. 1, 101–113.

M. D. Atkinson and S. J. van Willigenburg, *The p-modular descent algebra of the symmetric group*, Bull. London Math. Soc. **29** (1997), no. 4, 407–414.

D. J. Benson and K. J. Lim, *Projective modules and cohomology for integral basic algebras*, arXiv:2407.05531.

I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26.

References II

- T. P. Bidigare, *Hyperplane arrangement face algebras and their associated Markov chains*, Ph. d. dissertation, University of Michigan, 1997.
- K. Erdmann and K. J. Lim, *Representation type of the descent algebras of type* $\mathbb{A}$, arXiv:2407.06471.
- P. N. Norton, 0-*Hecke algebras*, J. Austral. Math. Soc. **27** (1979), no. 3, 337–357.
- F. V. Saliola, *On the quiver of the descent algebra*, J. Algebra **320** (2008), no. 11, 3866–3894.

References III

M. Schocker, *The descent algebra of the symmetric group*, Representations of finite dimensional algebras and related topics in Lie theory and geometry (V. Dlab and C. M. Ringel, eds.), Fields Inst. Commun. vol. 40, American Math. Society, 2004, pp. 145–161.

L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), no. 2, 255–264.