Descent Algebras of Type A

Kay Jin Lim

Nanyang Technological University (NTU)

April 20, 2022

- $\mathcal{O}$ commutative ring with 1
- F is an algebraically closed field with characteristic p (either $p > 0$ or $p = \infty$ (or 0))
- A composition is a finite sequence of positive integers
- A partition is a non-increasing composition
- Composition of permutations is read from left to right: $(1, 2)(2, 3) = (1, 3, 2)$

Descent algebras of the Coxeter groups

Let (W, S) be a finite Coxeter system, that is, W is generated by S subject to the relations

$$(ss')^{m(s,s')} = 1$$

where $m(s, s')$ are positive integers such that $m(s, s) = 1$. For each subset $J \subseteq S$, let

$$\begin{aligned} W_J &= \text{parabolic subgroup generated by } J, \\ X_J &= \text{distinguished left transversals consisting of minimal} \\ &\quad \text{length elements for } W/W_J. \end{aligned}$$

Define

$$x_J = \sum_{w \in X_J} w \in \mathcal{O}W.$$

For subsets $J, K \subseteq S$, let X_{JK} be the distinguished double coset representatives of (W_J, W_K) in W.

Theorem 1 (Solomon 1976)

Let $\mathscr{D}_{\mathcal{O}}(W) = \mathrm{span}\{x_J : J \subseteq S\} \subseteq \mathcal{O}W$. *Then* $\mathscr{D}_{\mathcal{O}}(W)$ *is a subalgebra of* $\mathcal{O}W$ *with* $\mathcal{O}$*-rank* $2^{|S|}$ *where*

$$x_J x_K = \sum_{L \subseteq S} a_{JK}^L x_L$$

where a_{JK}^L *is the number of elements* $x \in X_{JK}$ *such that* $x^{-1} W_J x \cap W_K = W_L$.

- The algebra $\mathscr{D}_{\mathcal{O}}(W)$ is now known as the Solomon's descent algebra.
- It is usually not commutative.

In the type A case, i.e., W is the finite symmetric group, say $W = \mathfrak{S}_n$, and let us take

$$S = \{s_i := (i, i+1) : i \in [1, n-1]\}.$$

$$\text{subsets of } S \Leftrightarrow \text{compositions of } n$$
$$\text{parabolic subgroups} \Leftrightarrow \text{Young subgroups}$$

Using this notation, for each composition δ of n, if $J \subseteq S$ corresponds to δ, we write

$$\Xi^\delta = x_J.$$

The descent algebra is denoted as $\mathscr{D}_{\mathcal{O}}(n)$.

For an example, in terms of one-line form,

$$\Xi^{(2,2)} = 1234 + 1324 + 1423 + 2314 + 2413 + 3412,$$

which the summands correspond to row standard tableaux of shape $(2, 2)$.

Theorem 2 (Garsia-Reutenauer 1989)

Let δ, η be compositions of n. Then

$$\Xi^{\delta}\Xi^{\eta} = \sum_{\rho} a^{\rho}_{\delta,\eta}\Xi^{\rho}$$

where $a^{\rho}_{\delta,\eta}$ is the total number of matrices with natural number entries, row sums equal to δ, column sums equal to η and ρ is the composition (omitting zeroes) obtained by reading the entries of the matrices row-wise from left to right and then top to bottom.

$$\Xi^{(2,2)}\Xi^{(2,1,1)} = \Xi^{(2,1,1)} + \Xi^{(1,1,2)} + 2\Xi^{(1,1,1,1)}$$

Mackey's formula for Young characters

If a composition δ can be rearranged to obtain another composition η, we write $\delta \approx \eta$.

For a composition δ of n, let φ^δ be the Young character, i.e., the character of the permutation module $M_{\mathbb{Z}}^\delta := \mathrm{ind}_{\mathfrak{S}_\delta}^{\mathfrak{S}_n}\mathbb{Z}$ where $\mathfrak{S}_\delta$ is the Young subgroup with respect to δ.

$$\mathfrak{S}_{(2,3,1)} = \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_1$$

If $\delta \approx \eta$, then $\varphi^\delta = \varphi^\eta$.

We have

$$\varphi^{\delta}\varphi^{\eta} = \sum_{\rho} a^{\rho}_{\delta,\eta}\varphi^{\rho}$$

where $a^{\rho}_{\delta,\eta}$ is the number given in Theorem 2.

Let $\mathscr{C}_n$ be the F-span of the F-valued Young characters, that is

$$\varphi^{\delta,F}(\mu) := \varphi^{\delta}(\mu) \cdot 1_F.$$

So $\mathscr{C}_n$ is a commutative F-algebra.

Let $n = 4$ and $p = 3$.

	(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
(4)	1	1	1	1	1
$(3,1)$	0	1	0	2	4
$(2,2)$	0	0	2	2	6
$(2,1,1)$	0	0	0	2	12
(1^4)	0	0	0	0	24

$\downarrow$ mod 3

	(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
(4)	1	1	1	1	1
$(3,1)$	0	1	0	2	1
$(2,2)$	0	0	2	2	0
$(2,1,1)$	0	0	0	2	0
(1^4)	0	0	0	0	0

$(3,1)$ and (1^4) belong to the same 3-equivalent class

Solomon's epimorphism

For a composition δ, let $\lambda(\delta)$ be the partition obtained from δ by rearrangement of its parts.

A p-regular partition is a partition without p parts of equal size. So any partition is ∞-regular.

Theorem 3 (Solomon 1976, Atkinson-van Willigenburg 1997)

The F-linear map

$$c_{n,F} : \mathscr{D}_F(n) \to \mathscr{C}_n$$

given by $c_{n,F}(\Xi^\delta) = \varphi^{\delta,F}$ is a surjective F-algebra homomorphism and $\ker(c_{n,F}) = \mathrm{rad}(\mathscr{D}_F(n))$ has an F-basis consisting of Ξ^δ such that $\lambda(\delta)$ is not p-regular and $\Xi^\delta - \Xi^\eta$ such that $\eta \neq \delta \approx \eta$.

Let δ be a composition of n and $L(\delta)$ be the last part of δ. For any strong refinement η of δ, i.e., η is the concatenation of $\eta^{(1)}, \ldots, \eta^{(\ell)}$ where ℓ is the length (total number of parts) of δ and each $\eta^{(i)}$ is a composition of δ_i, define

$$L_\delta(\eta) = \prod_{i=1}^{\ell} L(\eta^{(i)}).$$

$$L_{(2,3,4)}((2,2,1,2,2)) = 4$$

Let $m_i(\delta)$ be the number of parts of δ equal to i. Define

$$\delta\text{¡} = \prod_{i=1}^{\infty} m_i(\delta)!,$$

$$\delta? = \prod_{i=1}^{\infty} m_i(\delta)!\, i^{m_i(\delta)}.$$

For example,

$$(n)\text{¡} = 1, \qquad\qquad (1^n)\text{¡} = n!.$$

Notice that a partition λ is p-regular if and only if $\lambda\text{¡} \neq 0$ in F.

Define

$$\omega_\delta = \sum (-1)^{\ell(\eta)-\ell(\delta)} L_\delta(\eta)\Xi^\eta \in \mathscr{D}_{\mathcal{O}}(n)$$

where the sum runs over all strong refinements η of δ.

$$\omega_{(2,2)} = 4\Xi^{(2,2)} - 2\Xi^{(1,1,2)} - 2\Xi^{(2,1,1)} + \Xi^{(1,1,1,1)}$$

Theorem 4 (Blessenohl-Laue 1996)

(a) *The set* $\{\omega_\delta : \delta$ *a composition of* $n\}$ *is a basis for* $\mathscr{D}_{\mathbb{Q}}(n)$.

(b) $(\omega_\delta)^2 = \delta?\omega_\delta$ *in* $\mathscr{D}_{\mathcal{O}}(n)$

Complete set of primitive orthogonal idempotents of $\mathscr{D}_{\mathbb{Q}}(n)$

- The primitive orthogonal idempotents of $\mathscr{D}_{\mathbb{Q}}(n)$ are labelled by partitions of n.
- [Garsia-Reutenauer 1989] Explicit presentation in terms of Ξ^{δ}'s.
- [Blessenohl-Laue 1996] Orthogonalization of the (Lie) idempotents

$$\left\{ \frac{1}{\lambda ?}\omega_\lambda : \lambda \text{ a partition of } n \right\}.$$

In positive characteristic case, the primitive orthogonal idempotents of $\mathscr{D}_F(n)$ are labelled by p-regular partitions of n (Erdmann-Schocker 2006).

Let $\Lambda_p^+(n)$ denote the set of p-regular partitions of n and $\mathrm{char}_{\lambda,F} \in \mathscr{C}_n$ be the characteristic function on the p-equivalent class labelled by λ.

Theorem 5 (L. 2021)

We have a complete set of primitive orthogonal idempotents $\{e_{\lambda,F} : \lambda \in \Lambda_p^+(n)\}$ *of* $\mathscr{D}_n$ *such that* $c_{n,F}(e_{\lambda,F}) = \mathrm{char}_{\lambda,F}$*,* $\sum_{\lambda \in \Lambda_p^+(n)} e_{\lambda,F} = 1$ *and*

$$e_{\lambda,F} = \frac{1}{\lambda¡}\Xi^\lambda + \epsilon_\lambda$$

where ϵ_λ *is a linear combination of some* Ξ^ξ *such that* ξ *is a strict (that is* $\xi \not\approx \lambda$*) weak refinement of* λ*.*

A glance into the construction

$$\mathscr{D}_F(n)/\mathrm{rad}(\mathscr{D}_F(n)) \cong \mathscr{C}_n$$

1. $\mathscr{C}_n$ has orthogonal idempotents $\{\mathrm{char}_{\lambda,F} : \lambda \in \Lambda_p^+(n)\}$
2. Construct elements f_λ in $\mathscr{D}_F(n)$ using the inverse matrix Φ^{-1} such that $c_{n,F}(f_\lambda) = \mathrm{char}_{\lambda,F}$, where Φ is the modulo p table Φ for Young characters of $\mathfrak{S}_n$
3. Use the idempotent lifting idea to obtain the desired idempotents of $\mathscr{D}_F(n)$ from $\{f_\lambda : \lambda \in \Lambda_p^+(n)\}$

Free Lie algebra

Let V be a finite dimensional vector space and $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra. It is a Lie algebra with the Lie bracket

$$[v, w] = v \otimes w - w \otimes v$$

for all $v, w \in T(V)$.

Let $L(V)$ be the free Lie subalgebra of $T(V)$ generated by V and let

$$L^n(V) = L(V) \cap V^{\otimes n}$$

be the nth Lie power of V.

Let V be a right FG-module. Then $V^{\otimes n}$ is $(F\mathfrak{S}_n\text{-}FG)$-bimodule with the Pólya action of $\mathfrak{S}_n$ on the left. So

$$L^n(V) = \omega_n \cdot V^{\otimes n}$$

is a (right) FG-module where $\omega_n = \omega_{(n)}$ is the Dynkin-Specht-Wever element.

Question 6

Pick any finite dimensional FG-module V. Understand $L^n(V)$.

In the case $G = \mathrm{GL}(V)$, applying the Schur functor f, we get the Lie module

$$\mathrm{Lie}(n) = f(L^n(V)) \cong \omega_n F\mathfrak{S}_n.$$

History of study of the nth Lie powers

- [Witt 1937] $\dim_F L^n(V)$
- [Thrall 1942, Brandt 1944, Wever 1949, Klyachko 1974, Kraśkiewicz-Weyman 2011] $p = 0$ and $G = \mathrm{GL}(V)$
- [Hall 1950, Lyndon 1954] bases for $L(V)$
- [Donkin-Erdmann 1998] $p > 0$, $p \nmid n$ and $G = \mathrm{GL}(V)$
- [Bryant-Kovács-Stöhr 2002] $p > 0$, $G = C_p$
- [Bryant-Stöhr 2005] $p > 0$, $n = p$, $G = \mathrm{GL}(V)$
- [Erdmann-Schocker 2006] $p > 0$, $n = pk$, $p \nmid k$, $G = \mathrm{GL}(V)$

Higher Lie powers

For a composition δ, define the higher Lie power

$$L^{\delta}(V) = \omega_{\delta} \cdot V^{\otimes n}.$$

In the case $G = \mathrm{GL}(V)$, applying the Schur functor f, we get the higher Lie module

$$\mathrm{Lie}(\delta) = f(L^{\delta}(V)) \cong \omega_{\delta} F \mathfrak{S}_n.$$

Study of the higher Lie powers:

- [Schocker 2003] $p = 0$, $G = \mathrm{GL}(V)$

A composition δ is coprime to p if every part of δ is not divisible by p. In this case, we write $(\delta, p) = 1$. For an example, $(\delta, 0) = 1$ for any composition δ (no part with size 0).

Theorem 7 (L. 2021)

Let V be a right FG-module and δ be a composition. Suppose that $(\delta, p) = 1$ and the partition $\lambda(\delta)$ obtained from δ by arrangement of its parts is p-regular. We have

$$L^{\delta}(V) \cong \bigotimes_{i=1}^{\infty} S^{m_i}(L^i(V))$$

where $m_i = m_i(\delta)$.

The assumptions, $(\delta, p) = 1$ and $\lambda(\delta)$ is p-regular, are vacuous when $p = 0$.

As corollaries of the previous theorem, when $(\delta, p) = 1$ and $\lambda(\delta)$ is p-regular, we knew

- the dimension of $L^\delta(V)$,
- when $G = \mathrm{GL}(V)$, the formal character of $L^\delta(V)$,
- when $G = \mathrm{GL}(V)$, the decomposition of $L^\delta(V)$ as a direct sum of tilting modules.

Without the assumption on $\lambda(\delta)$ in the previous theorem (only $(\delta, p) = 1$ is assumed), we only get a surjection

$$\bigotimes_{i=1}^{\infty} S^{m_i}(L^i(V)) \twoheadrightarrow L^{\delta}(V).$$

However, if $G = \mathrm{GL}(V)$ and we apply Schur functor f, we still get isomorphism

$$f\left(\bigotimes_{i=1}^{\infty} S^{m_i}(L^i(V))\right) \cong f(L^{\delta}(V)) = \mathrm{Lie}(\delta).$$

A connection between Lie modules and higher Lie modules

Theorem 8 (Erdmann-Schocker 2006)

Suppose that $p > 0$ and $p \nmid k$. We have a short exact sequence of right $F\mathfrak{S}_{kp}$-modules

$$0 \to \mathrm{Lie}(kp) \to e_{(kp)}F\mathfrak{S}_{kp} \to \mathrm{Lie}((k^p)) \to 0.$$

Since $e_{(kp)}F\mathfrak{S}_{kp}$ is projective (right) $F\mathfrak{S}_{kp}$-module, there is a one-to-one correspondence between the non-projective indecomposable summands of $\mathrm{Lie}(kp)$ and those of $\mathrm{Lie}((k^p))$.

Projectivity of the higher Lie modules

Complexity of a module M is a natural number measuring the projectivity of M. The complexity is 0 if and only if M is projective. The complexity is 1 if and only if M is periodic.

Theorem 9 (Cohen-Hemmer-Nakano 2016)

The complexity of the Lie module $\mathrm{Lie}(n)$ *is* s *where* $p^s \mid n$ *but* $p^{s+1} \nmid n$.

Theorem 10 (L. 2021)

When $(\delta, p) = 1$, the higher Lie module $\mathrm{Lie}(\delta)$ has complexity

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m_i(\delta)}{p} \right\rfloor .$$

Question 11

In general, what is the complexity of $\mathrm{Lie}(\delta)$?

Thank you!