Cores and Core Blocks of Ariki-Koike Algebras

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Throughout $e \in \mathbb{Z}_{\geq 2}$.

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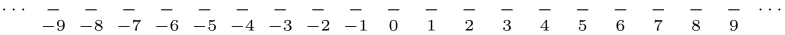
We obtain the **e-abacus display** of S by cutting up its ∞-abacus display into sections $[ie, ie + e - 1]$ ($i \in \mathbb{Z}$), and putting the section $[ie, ie + e - 1]$ directly on top of $[(i + 1)e, (i + 1)e + e - 1]$.

Example

...	_	_	_	_	_	_	_	_	_	_	_	_	_	_	_	_	_	_	_	...
	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	

Example



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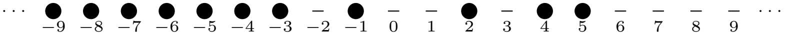

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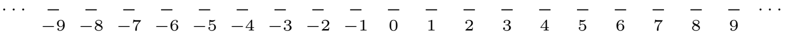

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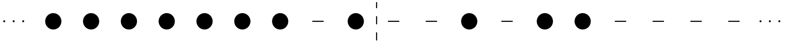

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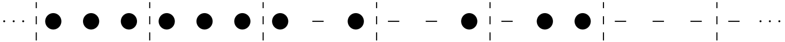

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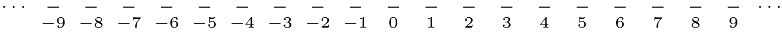

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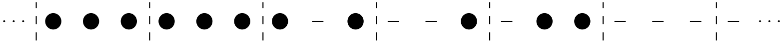
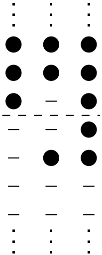


3-abacus display of B



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Indeed, $\beta_{\mathfrak{s}(B)}(\lambda) = B$.

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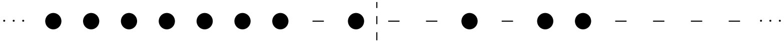

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e-Core and e-Weight of a Partition

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Note that $\mathrm{core}_e(\lambda)$ and $\mathrm{wt}_e(\lambda)$ are independent of the charge s.

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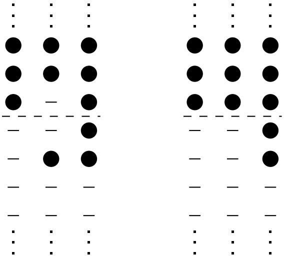

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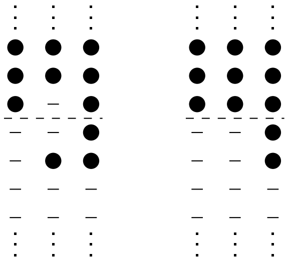

$\beta_2(\lambda)$ $\beta_2(\mathsf{core}_3(\lambda))$

$\mathsf{core}_3(\lambda) = (4, 2, 0, 0, 0, \dots) = (4, 2)$, and $\mathsf{wt}_3(\lambda) = 2$.

ℓ-partitions and the Uglov Map

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- Stack the ∞-abacus displays of $\beta_{s_i}(\lambda^{(i)})$ on top of each other, with the display of $\beta_{s_1}(\lambda^{(1)})$ at the bottom and that of $\beta_{s_\ell}(\lambda^{(\ell)})$ at the top.

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Example

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = ((1), (1,1), (2)),\ \mathbf{s} = (s_1, s_2, s_3) = (-1, 2, 1).$$

$\beta_{s_3}(\lambda^{(3)}) = \{2, -1, -2, -3\ldots\}$	$\cdots$	●	●	● ¦	–	–	●	–	–	–	$\cdots$
$\beta_{s_2}(\lambda^{(2)}) = \{2, 1, -1, -2, \ldots\}$	$\cdots$	●	●	● ¦	–	●	●	–	–	–	$\cdots$
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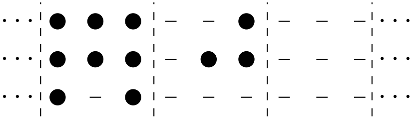

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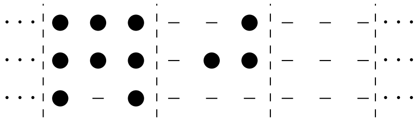
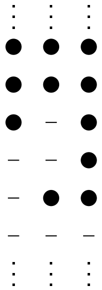


$\mathsf{U}_3(\boldsymbol{\lambda}; \mathbf{s})$



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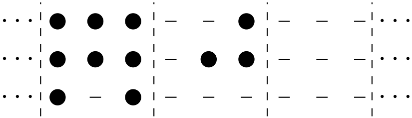
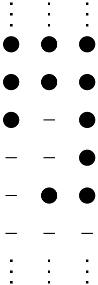


$\mathsf{U}_3(\boldsymbol{\lambda}; \mathbf{s}) = (4, 4, 3, 1)$



e-Core and e-Weight of a Multipartition

Definition

Let $\boldsymbol{\lambda}$ be an ℓ-partition and let s be an ℓ-charge. We define the e-core and the e-weight of $(\boldsymbol{\lambda};\mathrm{s})$ to be those of $\mathsf{U}_e(\boldsymbol{\lambda};\mathrm{s})$; i.e.

$$\mathsf{core}_e(\boldsymbol{\lambda};\mathrm{s}) = \mathsf{core}_e(\mathsf{U}_e(\boldsymbol{\lambda};\mathrm{s}));$$
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$$\mathsf{core}_3(((1),(1,1),(2));(-1,2,1)) = \mathsf{core}_3(4,4,3,1) = (4,2)$$
$$\mathsf{wt}_3(((1),(1,1),(2));(-1,2,1)) = \mathsf{wt}_3(4,4,3,1) \quad = 2.$$

The extended affine Weyl group $\widehat{\mathbf{W}}_\ell$

Given any nonempty set X, the symmetric group $\mathfrak{S}_\ell$ on ℓ letters has a natural right place permutation action on X^ℓ via

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This right action gives rise to the extended affine Weyl group $\widehat{\mathbf{W}}_\ell = \mathbb{Z}^\ell \rtimes \mathfrak{S}_\ell$, which has a natural right action on the pairs of ℓ-partitions and their respective associated ℓ-charges via

$$(\boldsymbol{\lambda}; \mathbf{s})^{\mathbf{t}\sigma} = (\boldsymbol{\lambda}^\sigma; (\mathbf{s} + e\mathbf{t})^\sigma) \qquad (\mathbf{t} \in \mathbb{Z}^\ell, \sigma \in \mathfrak{S}_\ell).$$

Let $\overline{\mathscr{A}_e^\ell} = \{(s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell : s_1 \leq s_2 \leq \cdots \leq s_\ell \leq s_1 + e\}$.

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Theorem (Li-T.)

Let $\boldsymbol{\lambda}$ be an ℓ-partition and $\mathbf{s}$ be an ℓ-charge. Let $(\boldsymbol{\mu}; \mathbf{t}) \in (\boldsymbol{\lambda}; \mathbf{s})^{\widehat{\mathbf{W}}_\ell}$, the $\widehat{\mathbf{W}}_\ell$-orbit of $(\boldsymbol{\lambda}; \mathbf{s})$.

1. $\mathsf{core}_e(\boldsymbol{\mu}; \mathbf{t}) = \mathsf{core}_e(\boldsymbol{\lambda}; \mathbf{s})$.
2. $\mathsf{wt}_e(\boldsymbol{\mu}; \mathbf{t}) = \min(\mathsf{wt}_e((\boldsymbol{\lambda}; \mathbf{s})^{\widehat{\mathbf{W}}_\ell}))$ *if and only if* $\mathbf{t} \in \overline{\mathscr{A}_e^\ell}$.

Ariki-Koike Algebras

Let $\mathbf{r} = (r_1, \ldots, r_\ell) \in \mathbb{Z}^\ell$ and $n \in \mathbb{Z}^+$.
Let $\mathbb{F}$ be a field of arbitrary characteristic, with a primitive e-th root of unity $q \in \mathbb{F}$.

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The Ariki-Koike algebra $\mathscr{H}_n = \mathscr{H}_{\mathbb{F},q,\mathbf{r}}(n)$ is the unital $\mathbb{F}$-algebra generated by $\{T_0, T_1, \ldots, T_{n-1}\}$ subject to:

$$
\begin{aligned}
(T_0 - q^{r_1})(T_0 - q^{r_2}) \cdots (T_0 - q^{r_\ell}) &= 0; \\
(T_i - q)(T_i + 1) &= 0 && (1 \le i \le n-1); \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0; \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && (1 \le i \le n-2); \\
T_i T_j &= T_j T_i && (|i-j| \ge 2).
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T_i T_j &= T_j T_i && (|i-j| \ge 2).
\end{aligned}$$

When $\ell = 1$, $\mathscr{H}_n$ is the Iwahori-Hecke algebra of type A.
When $\ell = 2$, $\mathscr{H}_n$ is the Iwahori-Hecke algebra of type B.

Blocks of $\mathscr{H}_n$

$\mathscr{H}_n$ is cellular (in the sense of Graham-Lehrer), with the **Specht modules** (constructed by Dipper-James-Mathas), indexed by the set of ℓ-partitions of n, as its cell modules.

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The elements of its Young diagram $[\boldsymbol{\lambda}] = \{(a, b, j) \in (\mathbb{Z}^+)^3 : j \leq \ell,\ a \leq \ell(\lambda^{(j)}),\ b \leq \lambda_a^{(j)}\}$ are called **nodes**.

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The residue of $(a,b,j) \in [\boldsymbol{\lambda}]$ is the residue class of $b - a + r_j$ modulo e, and (a,b,j) is called an i-**node** if its residue equals i.

Theorem (Lyle-Mathas 07)

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be ℓ-partitions of n. The Specht modules $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ lie in the same block of $\mathscr{H}_n$ if and only if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same number of i-nodes for all $i \in \mathbb{Z}/e\mathbb{Z}$.

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Theorem (James)

($\ell = 1$) The partitions λ and μ have the same number of i-nodes for all $i \in \mathbb{Z}/e\mathbb{Z}$ if and only if λ and μ have the same e-core and the same e-weight.

Weights of Multipartitions

Definition (Fayers 06)

Let $\boldsymbol{\lambda}$ be an ℓ-partition of n. Define

$$\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \sum_{j=1}^{\ell} c_{\overline{r_j}}(\boldsymbol{\lambda}) + \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i(\boldsymbol{\lambda}) - c_{i+1}(\boldsymbol{\lambda}))^2,$$

where $c_i(\boldsymbol{\lambda})$ is the number of i-nodes in $[\boldsymbol{\lambda}]$, and $\overline{r_j}$ is the residue class of r_j modulo e.

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Theorem (Fayers 06)

$\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda})$ *is a block invariant,* $\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$*, and when* $\ell = 1$*,* $\mathrm{wt}_{\mathscr{H}}(\lambda) = \mathrm{wt}_e(\lambda)$*.*

Theorem (Jacon-Lecouvey 21)

If $\mathbf{r} \in \overline{\mathscr{A}_e^{\ell}}$, *then*

$$\mathsf{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \mathsf{wt}_e(\mathsf{U}_e(\boldsymbol{\lambda};\mathbf{r})) \ (= \mathsf{wt}_e(\boldsymbol{\lambda};\mathbf{r})).$$

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Corollary (Li-T.)

$$\mathsf{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \min(\mathsf{wt}_e((\boldsymbol{\lambda};\mathbf{r})^{\widehat{\mathbf{W}}_\ell})).$$

Cores of Multipartitions

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Define $\mathrm{core}_{\mathscr{H}}(\boldsymbol{\lambda}) := \mathrm{core}_e(\boldsymbol{\lambda};\mathbf{r})$.

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Theorem (Nakayama's 'Conjecture' for Ariki-Koike algebras (Li-T.))

Two Specht modules $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ (possibly of different algebras) lie in the same block if and only if $\mathrm{core}_{\mathscr{H}}(\boldsymbol{\lambda}) = \mathrm{core}_{\mathscr{H}}(\boldsymbol{\mu})$ and $\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \mathrm{wt}_{\mathscr{H}}(\boldsymbol{\mu})$.

Cores and weights of a block

Definition

The **core** and the **weight** of a block B of $\mathscr{H}_n$, denoted $\mathrm{core}_{\mathscr{H}}(B)$ and $\mathrm{wt}_{\mathscr{H}}(B)$, are the common e-core and the common e-weight of the ℓ-partitions lying in B.

Action of the affine Weyl group $\mathbf{W}_e$

Let $\boldsymbol{\lambda}$ be an ℓ-partition. A **removable** node of $\boldsymbol{\lambda}$ is an $\mathfrak{n} \in [\boldsymbol{\lambda}]$ such that $[\boldsymbol{\lambda}] \setminus \{\mathfrak{n}\} = [\boldsymbol{\mu}]$ for some ℓ-partition $\boldsymbol{\mu}$; in which case $\mathfrak{n}$ is also called an **addable** node of $\boldsymbol{\mu}$.

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For each $i \in \mathbb{Z}/e\mathbb{Z}$, write $\mathsf{s}_i(\boldsymbol{\lambda})$ for the ℓ-partition obtained by removing all removable i-nodes of $\boldsymbol{\lambda}$ and adding all addable i-nodes of $\boldsymbol{\lambda}$. This induces a left action of the affine Weyl group $\mathbf{W}_e = \langle \mathsf{s}_i \mid i \in \mathbb{Z}/e\mathbb{Z} \rangle$ on the set of ℓ-partitions.

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This action preserves weights of ℓ-partitions, and $\mathrm{core}_{\mathscr{H}}(\mathsf{s}_i(\boldsymbol{\lambda})) = \mathsf{s}_i(\mathrm{core}_{\mathscr{H}}(\boldsymbol{\lambda}))$. Consequently, we get a left action of $\mathbf{W}_e$ on the set of blocks of Ariki-Koike algebras.

Scopes Equivalence

Theorem (Chuang-Rouquier 08)

If every ℓ-partition lying B has no removable i-node (or no addable i-node), then B and $\mathsf{s}_i(B)$ are Morita equivalent, with $S^{\boldsymbol{\lambda}} \leftrightarrow S^{\mathsf{s}_i(\boldsymbol{\lambda})}$ for all $\boldsymbol{\lambda}$ lying in B.

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We further extend Scopes equivalence to an equivalence relation on the set of blocks of Ariki-Koike algebras by taking its reflexive and transitive closure.

$[w : k]$-Pairs

Let B be a block of $\mathscr{H}_n$, and let $i \in \mathbb{Z}/e\mathbb{Z}$.
Assume that $\mathsf{s}_i(B)$ is a block of $\mathscr{H}_{n-k}$ with $k > 0$.

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Theorem (Li-T.)

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If B and $\mathsf{s}_i(B)$ form a $[w:k]$-pair with $k \geq w$, then B and $\mathsf{s}_i(B)$ are Scopes equivalent.
(The converse is false.)

Core blocks

Definition (Fayers 07)

A block of an Ariki-Koike algebra is a **core block** if every multipartition lying in it is a multicore.

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Remark

1. It is not true that every multicore lies in a core block.
2. The weight of a core block can be arbitrarily big if there is no restriction on e and ℓ.

Moving vectors of multipartitions

Given an ℓ-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ with an associated ℓ-charge $\mathbf{s}$, define its moving vector $\mathsf{mv}_e(\boldsymbol{\lambda}; \mathbf{s}) = (m_1, \dots, m_\ell)$ as follows:

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Label the rows of the e-abacus display of $\mathsf{U}_e(\boldsymbol{\lambda}; \mathbf{s})$ by $\{1, \dots, \ell\}$ according to which ∞-abacus displays of $\lambda^{(i)}$'s they come from. Then m_i is the number of times a bead from a row labelled by i is moved to the row immediately above it when the beads in the e-abacus display of $\mathsf{U}_e(\boldsymbol{\lambda}; \mathbf{s})$ are moved to obtained its e-core.

Example

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = ((1), (1,1), (2)),\ \mathbf{s} = (s_1, s_2, s_3) = (-1, 2, 1).$$

$\beta_{s_3}(\lambda^{(3)}) = \{2, -1, -2, -3 \dots\}$	$\cdots$	●	●	●	–	–	●	–	–	–	$\cdots$
$\beta_{s_2}(\lambda^{(2)}) = \{2, 1, -1, -2, \dots\}$	$\cdots$	●	●	●	–	●	●	–	–	–	$\cdots$
$\beta_{s_1}(\lambda^{(1)}) = \{-1, -3, -4, \dots\}$	$\cdots$	●	–	●	–	–	–	–	–	–	$\cdots$

$\mathsf{U}_3(\boldsymbol{\lambda}; \mathbf{s}) = (4, 4, 3, 1)$

$\vdots$	$\vdots$	$\vdots$
●	●	●
●	●	●
●	–	●
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$$\begin{array}{rc|ccc|ccc|ccc|c}
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\end{array}$$

$$\begin{array}{rcccl}
 & \vdots & \vdots & \vdots & \\
 & \bullet & \bullet & \bullet & 3 \\
 & \bullet & \bullet & \bullet & 2 \\
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Thus $\mathsf{mv}_3(\boldsymbol{\lambda}; \mathbf{s}) = (0, 1, 1)$.

Moving vectors of blocks of Ariki-Koike algebras

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The moving vector is a block invariant; that is, ℓ-partitions having the same core and same weight have the same moving vector, so one can talk about the moving vector $\mathrm{mv}(B)$ of a block B of an Ariki-Koike algebra.

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This notion of moving vectors is first introduced by Yanbo Li and Xiangyu Qi to study the representation types of the blocks of Ariki-Koike algebras.

Moving vectors of core blocks

Theorem (Li-Qi-T.)

Let B be a block of $\mathscr{H}_n$, with $\mathsf{mv}(B) = (m_1, \ldots, m_\ell)$*. Then B is a core block if and only if $m_j = 0$ for some j.*

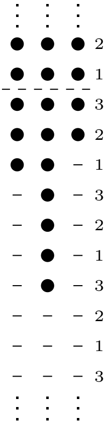
Scopes vectors

To a given core block B with $\mathsf{mv}(B) = (m_1, \ldots, m_\ell)$ and $m_j = 0$, we can associate a j-**Scopes vector** $\mathbf{Sc}_j(B) \in \{0, 1, 2, \ldots, \ell - 1\}^e$.

Example

Let $e = \ell = 3$, $\mathbf{r} = (3,3,6)$, $\mathsf{core}_3(B) = (8,6,4,2)$, $\mathsf{mv}(B) = (1,0,1)$.

$\beta_{12}(\mathsf{core}_3(B))$



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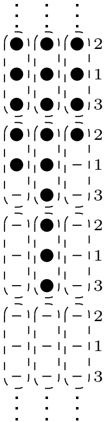

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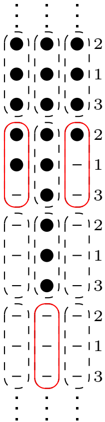
$\beta_{12}(\mathsf{core}_3(B))$ 

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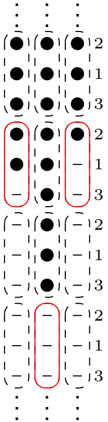

Thus $\mathbf{Sc}_2(B) = (2, 1, 0)$.

Scopes equivalence between core blocks

Theorem (Li-Qi-T.)

Let B and B' be core blocks of Ariki-Koike algebras (with the same associated ℓ-charge), with the same moving vector $(m_1, \dots, m_\ell)$ where $m_j = 0$. If $\mathbf{Sc}_j(B) = \mathbf{Sc}_j(B')$, then B and B' are Scopes equivalent.

Corollary

Let B be a core block of $\mathscr{H}_n$. Then

$$\mathsf{wt}_e(B) \leq \left\lfloor \frac{\ell}{2} \right\rfloor \left\lceil \frac{\ell}{2} \right\rceil e.$$