Young's Seminormal Basis Vectors and Their Denominators

Kai Meng Tan

National University of Singapore

Joint work with Ming Fang (Chinese Academy of Sciences)
and Kay Jin Lim (Nanyang Technological University)

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Partitions and Young diagrams

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Example

$\lambda = (5, 4, 4, 2, 1)$.

$[\lambda] =$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)	(2,2)	(2,3)	(2,4)	
(3,1)	(3,2)	(3,3)	(3,4)	
(4,1)	(4,2)			
(5,1)				

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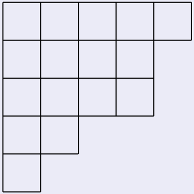
$[\lambda] =$ 

A **node** (i,j) of $[\lambda]$ is **removable** if $(i+1,j),(i,j+1) \notin [\lambda]$. For example, the removable nodes of $[5,4,4,2,1]$ are $(1,5),(3,4),(4,2),(5,1)$.

Young tableaux

Let λ be a partition of n. A λ-tableau $\mathfrak{t}$ is formally a bijective function $\mathfrak{t} : [\lambda] \to \{1, \ldots, n\}$, which we view as a filling of the boxes in $[\lambda]$ by $1, \ldots, n$ so that each number appears once.

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A λ-tableau is **standard** if its entries increase along each row (from left to right) and down each column. Denote the set of all standard λ-tableaux by $\mathrm{Std}(\lambda)$, and let $\hbar^\lambda = |\mathrm{Std}(\lambda)|$.

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The **initial λ-tableau** $\mathfrak{t}^\lambda$ is the standard λ-tableau obtained by filling in $1, \ldots, \lambda_1$ in the first row of $[\lambda]$, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row of $[\lambda]$, and so on.

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The symmetric group $\mathfrak{S}_n$ acts on the set of λ-tableaux of n from the left by post-composition (or permuting on the numbers in the boxes of $[\lambda]$). We may thus speak of the row stabilizer $R_\mathfrak{t}$ and the column stabilizer $C_\mathfrak{t}$ of a λ-tableau $\mathfrak{t}$.

Young symmetrizers and dual Specht modules

Let λ be a partition of n. For a λ-tableau $\mathfrak{t}$, its associated Young symmetrizer is

$$e_{\mathfrak{t}} = \sum_{\rho \in R_{\mathfrak{t}}} \sum_{\kappa \in C_{\mathfrak{t}}} \mathrm{sgn}(\kappa)\rho\kappa \in \mathbb{Z}\mathfrak{S}_n.$$

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Since $\sigma e_{\mathfrak{t}} = e_{\sigma \cdot \mathfrak{t}}$ for any $\sigma \in \mathfrak{S}_n$, the R-span of the Young symmetrizers associated to λ-tableaux is a left ideal of $R\mathfrak{S}_n$ for any commutative ring R with 1. This is the dual Specht module S_λ^R.

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Furthermore $\{e_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\}$ is a R-basis for S_λ^R, called the **standard basis**.

Residues

Let λ be a partition of n. The **residue** of a node $A = (i, j) \in [\lambda]$, denoted $\mathrm{res}(A)$, is $j - i$.

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If $\mathfrak{t}$ is a λ-tableau and $1 \leq i \leq n$, then the residue of i in $\mathfrak{t}$, denoted $\mathrm{res}_{\mathfrak{t}}(i)$, is the residue of the node in which i appears in $\mathfrak{t}$. More formally, $\mathrm{res}_{\mathfrak{t}}(i) = \mathrm{res}(\mathfrak{t}^{-1}(i))$.

Young's seminormal basis

The dual Specht module $S_\lambda^{\mathbb{Q}}$ has another distinguished basis $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$, called **Young's seminormal basis**, which has the following characterising property:

1. $f_{\mathfrak{t}^\lambda} = e_{\mathfrak{t}^\lambda}$;
2. $f_{\mathsf{s}_i \cdot \mathfrak{t}} = (\mathsf{s}_i + \frac{1}{\mathrm{res}_{\mathfrak{t}}(i) - \mathrm{res}_{\mathfrak{t}}(i+1)}) f_{\mathfrak{t}}$, where $\mathsf{s}_i = (i, i+1)$, whenever $\mathfrak{t}, \mathsf{s}_i \cdot \mathfrak{t} \in \mathrm{Std}(\lambda)$ with $\mathrm{res}_{\mathfrak{t}}(i) > \mathrm{res}_{\mathfrak{t}}(i+1)$.

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The **denominator** of $f_{\mathfrak{t}}$, denoted $\mathsf{d}_{\mathfrak{t}}$, is the least $k \in \mathbb{Z}^+$ such that $k f_{\mathfrak{t}}$ lies in the $\mathbb{Z}$-span of the standard basis.

The tableau $\mathfrak{t}^{\lambda\uparrow^\nu}$

Let λ and ν be partitions of n and m respectively, such that $[\lambda] \subseteq [\nu]$. Define the ν-tableau $\mathfrak{t}^{\lambda\uparrow^\nu}$ to be that obtained by filling in the nodes of $[\lambda]$ according to $\mathfrak{t}^\lambda$ and the nodes of $[\nu] \setminus [\lambda]$ with $[n+1, m]$ in turn, starting with the top row, going from left to right in each row, and down the rows.

The tableau $t^{\lambda\uparrow^\nu}$

Let λ and ν be partitions of n and m respectively, such that $[\lambda] \subseteq [\nu]$. Define the ν-tableau $t^{\lambda\uparrow^\nu}$ to be that obtained by filling in the nodes of $[\lambda]$ according to t^λ and the nodes of $[\nu] \setminus [\lambda]$ with $[n+1, m]$ in turn, starting with the top row, going from left to right in each row, and down the rows.

Example

$\lambda = (3, 2, 2)$ and $\nu = (5, 4, 4, 2, 1)$.

$t^{\lambda\uparrow^\nu} =$

1	2	3	8	9
4	5	10	11	
6	7	12	13	
14	15			
16				

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Furthermore, we write $\mathfrak{t}^{\lambda|m}$ for $\mathfrak{t}^{\lambda\uparrow^{\lambda+(m)}}$ for $m \in \mathbb{Z}^+$.

Motivation

Theorem (Fang-Lim-T.)

If the last column of $[\lambda]$ is not shorter than the first column of $[\mu]$, and $p \nmid \frac{\hbar^{\lambda+\mu} \mathsf{d}_{\mathfrak{t}^{\lambda} \uparrow \lambda+\mu}}{\hbar^{\lambda}\hbar^{\mu}}$, then the Jantzen filtration of $S_{\lambda}^{\mathbb{Z}_{(p)}}$ may be 'naturally embedded' into that of $S_{\lambda+\mu}^{\mathbb{Z}_{(p)}}$.

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Can we compute $\mathsf{d}_{\mathfrak{t}^{\lambda} \uparrow^{\lambda+\mu}}$ easily?

Each Young's seminormal basis vector $f_{\mathfrak{t}}$ may be expressed in terms of the standard basis using the recursive characterising relation. But there is no known closed formula in general.

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Each Young's seminormal basis vector $f_{\mathfrak{t}}$ may be expressed in terms of the standard basis using the recursive characterising relation. But there is no known closed formula in general.

In this talk, we study $f_{\mathfrak{t}^{\lambda}\uparrow^{\nu}}$ and its denominator.

Semistandard tableaux

Let λ and ν be partitions with $[\lambda] \subseteq [\nu]$. Let $\nu_\lambda = (\lambda, \nu - \lambda)$.

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An integer i is said to be of colour j if i appears in the j-th row of $\mathfrak{t}^{\nu_\lambda}$.

A standard ν-tableau $\mathfrak{s}$ is (ν_λ-)semistandard, or simply **sstd**, if the colours of the integers appearing in each column of $\mathfrak{s}$ are distinct.

Example

Let $\lambda = (3,3,2,2)$ and $\nu = (4,3,2,2)$ so that $\nu_\lambda = (3,3,2,2,1)$. The following ν-tableaux are sstd:

1	2	3	6
4	5	11	
7	8		
9	10		

1	2	3	10
4	5	6	
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The following standard tableaux on the other hand are not sstd:

1	2	3	11
4	5	7	
6	8		
9	10		

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4	7	8	
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Theorem

Let λ and ν be partitions with $[\lambda] \subseteq [\nu]$. Let

$$f_{\mathfrak{t}^{\lambda} \uparrow^{\nu}} = \sum_{\mathfrak{s} \in \mathrm{Std}(\lambda)} q_{\mathfrak{s}} e_{\mathfrak{s}}.$$

1. *If $\mathfrak{s}$ is not sstd, then $q_{\mathfrak{s}} = 0$.*

Theorem

Let λ and ν be partitions with $[\lambda] \subseteq [\nu]$. Let

$$f_{\mathfrak{t}^{\lambda\uparrow\nu}} = \sum_{\mathfrak{s}\in \mathrm{Std}(\lambda)} q_{\mathfrak{s}} e_{\mathfrak{s}}.$$

1. *If $\mathfrak{s}$ is not sstd, then $q_{\mathfrak{s}} = 0$.*
2. *If $\mathfrak{s}(i,j)$ and $\mathfrak{s}'(i,j)$ have the same colour for all $(i,j) \in [\nu]$, then $q_{\mathfrak{s}} = q_{\mathfrak{s}'}$.*

The vector $f_{\mathfrak{t}^{\lambda|1}}$

For each sstd tableau $\mathfrak{s}$, define

$$P(\mathfrak{s}) := \{i \geq 2 \mid \mathfrak{s} \text{ has an integer in row i with colour} \neq i \wedge (\lambda_i \neq \lambda_{i-1} \vee \text{all integers in row } i-1 \text{ of } \mathfrak{s} \text{ have colour } i-1)\}.$$

Furthermore, for each $i \in P(\mathfrak{s})$, define $m_i := \max\{j \mid \lambda_j = \lambda_i\}$.

Theorem

$$f_{\mathfrak{t}^{\lambda|1}} = \sum_{\text{sstd } \mathfrak{s}} \left(\prod_{i \in P(\mathfrak{s})} \frac{(-1)^{m_i - i}}{\lambda_1 - \lambda_i + m_i} \right) e_{\mathfrak{s}}.$$

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Corollary

Let $A_0, A_1, \ldots, A_t$ be all the removable nodes of $\lambda + (1)$, labelled from top to bottom. Then

$$\mathsf{d}_{\mathfrak{t}^{\lambda|1}} = \prod_{i=1}^{t} (\mathrm{res}(A_0) - \mathrm{res}(A_i)).$$

The vector $f_{\mathfrak{t}(k,\ell)|m}$

For each sstd tableau $\mathfrak{s}$, define $\mathrm{wt}(\mathfrak{s})$ to be the number of integers with colour 2 in the first row of $\mathfrak{s}$.

Theorem

$$f_{\mathfrak{t}(k,\ell)|m} = \sum_{\text{sstd } \mathfrak{s}} \frac{1}{\binom{k-\ell+1+\mathrm{wt}(\mathfrak{s})}{\mathrm{wt}(\mathfrak{s})}} e_{\mathfrak{s}}.$$

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Corollary

$$\mathsf{d}_{\mathfrak{t}(k,\ell)|m} = \frac{\mathrm{lcm}[k-\ell+1,\ k-\ell+1+\min(\ell,m)]}{k-\ell+1}.$$

The vector $f_{\mathfrak{t}(k,1^s)|m}$

For each sstd tableau $\mathfrak{s} \neq \mathfrak{t}^{(k,1^s|m)}$, let $\mathrm{wt}(\mathfrak{s}) = j$ if $\mathfrak{s}$ has an integer with colour $j+1$ in its first row.

Theorem

$$f_{\mathfrak{t}(k,1^s)|m} = e_{\mathfrak{t}(k,\ell)|m} + \frac{1}{k+s} \sum_{\substack{\text{sstd } \mathfrak{s} \\ \mathfrak{s} \neq \mathfrak{t}^{(k,\ell)|m}}} (-1)^{s-\mathrm{wt}(\mathfrak{s})} e_{\mathfrak{s}}.$$

Corollary

$$\mathsf{d}_{\mathfrak{t}(k,1^s)|m} = k+s.$$

The vector $f_{\mathfrak{t}^{(k,\ell^s)|m}}$

Let

$$f_{\mathfrak{t}^{(k,\ell^s)|m}} = \sum_{\text{sstd }\mathfrak{s}} q_{\mathfrak{s}}^{(k,\ell^s)|m} e_{\mathfrak{s}}.$$

For each sstd tableau $\mathfrak{s}$, define its **weight** $\mathrm{wt}(\mathfrak{s}) = (n_1(\mathfrak{s}), \ldots, n_s(\mathfrak{s}))$, where $n_j(\mathfrak{s})$ is the number of integers with colour $j+1$ in the first row of $\mathfrak{s}$.

The vector $f_{\mathfrak{t}(k,\ell^s)|m}$ (cont'd)

Theorem

1. *If* $\mathrm{wt}(\mathfrak{s}) = \mathrm{wt}(\mathfrak{s}')$, *then* $q_{\mathfrak{s}}^{(k,\ell^s)|m} = q_{\mathfrak{s}'}^{(k,\ell^s)|m}$. *Thus we may define* $q_{\mathrm{wt}(\mathfrak{s})}^{(k,\ell^s)|m}$ *without any ambiguity.*

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2. *The symmetric group* $\mathfrak{S}_s$ *acts naturally (from the left) on the set of weights* $\{\mathrm{wt}(\mathfrak{s}) \mid \mathfrak{s}$ *sstd*$\}$ *by place permutation. Then*

$$q_{\mathrm{wt}(\mathfrak{s})}^{(k,\ell^s)|m} = q_{\sigma\cdot\mathrm{wt}(\mathfrak{s})}^{(k,\ell^s)|m}$$

for all $\sigma \in \mathfrak{S}_s$.

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$$q_{\mathrm{wt}(\mathfrak{s})}^{(k,\ell^s)|m} = q_{\sigma\cdot\mathrm{wt}(\mathfrak{s})}^{(k,\ell^s)|m}$$

   for all $\sigma \in \mathfrak{S}_s$.
3.

$$q_{\mathbf{w}}^{(k,\ell^s)|m} = \begin{cases} q_{(0,\mathbf{w})}^{(k-1,\ell^{s+1})|m}; \\ q_{\mathbf{w}}^{(k,\ell^s)|m-1}, & \text{if } m > \ell; \\ q_{\mathbf{w}}^{(k-1,(\ell-1)^s)|m}, & \text{if } m < \ell. \end{cases}$$

The vector $f_{\mathfrak{t}(k,\ell^s)|m}$ (cont'd)

Corollary

Let $\tilde{\ell} = \min(\ell, m)$, $\tilde{k} = k - \ell + \max(s, \tilde{\ell})$, $\tilde{s} = \min(s, \tilde{\ell})$. *Then* $\tilde{k} \geq \tilde{\ell} \geq \tilde{s}$, *and*

$$\mathsf{d}_{\mathfrak{t}(k,\ell^s)|m} = \mathsf{d}_{\mathfrak{t}(\tilde{k},\tilde{\ell}^{\tilde{s}})|\tilde{\ell}}.$$

Denominators: Reduction

Theorem

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ *and* $\nu = (\nu_1, \dots, \nu_t)$ *be partitions with* $[\lambda] \subseteq [\nu]$.

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2. *If* $\lambda_1 = \nu_1$ *and* $r \geq 2$*, then* $\mathsf{d}_{\mathfrak{t}^{\lambda\uparrow\nu}} = \mathsf{d}_{\mathfrak{t}^{(\lambda_2,\ldots,\lambda_r)\uparrow(\nu_2,\ldots,\nu_t)}}$.

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3. *For* $1 \leq m \leq \nu_1 - \lambda_1$*,*

$$\mathsf{d}_{\mathfrak{t}^{\lambda \uparrow \nu}} \mid \mathsf{d}_{\mathfrak{t}^{\lambda | m}} \mathsf{d}_{\mathfrak{t}^{\lambda + (m) \uparrow \nu}}.$$

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3. *For* $1 \leq m \leq \nu_1 - \lambda_1$,
$$\mathsf{d}_{\mathfrak{t}^{\lambda \uparrow \nu}} \mid \mathsf{d}_{\mathfrak{t}^{\lambda | m}} \mathsf{d}_{\mathfrak{t}^{\lambda + (m) \uparrow \nu}}.$$
4. *For* $2 \leq i \leq r-1$ *and* $m \in \mathbb{Z}^+$,
$$\mathsf{d}_{\mathfrak{t}^{\lambda | m}} \mid \mathsf{d}_{\mathfrak{t}^{(\lambda_1 + i - 1, \lambda_{i+1}, \ldots, \lambda_r) | m}} \mathsf{d}_{\mathfrak{t}^{(\lambda_1, \ldots, \lambda_i) | m}}.$$

Corollary

1. *If λ is obtained from ν by removing a removable node A, and $B_1, \dots, B_s$ are removable nodes of ν below A, then*

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2. *For $k \geq \ell \geq s$,*

$$\mathsf{d}_{\mathfrak{t}^{(k,\ell^s)|\ell}} \mid \gcd \left(\prod_{i=1}^{\ell} (k - \ell + s + i),\ \prod_{j=1}^{s} \frac{\mathrm{lcm}[k - \ell + j,\ k + j]}{k - \ell + j} \right).$$

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3. *If $s \geq 2 = \min(\ell, m)$, then*

$$\mathsf{d}_{\mathfrak{t}(k,\ell^s)|m} = (k - \ell + s + 1)(k - \ell + s + 2).$$

Thank you for your attention!