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A SCHUR-POSITIVITY CONJECTURE INSPIRED BY THE ALPERIN-MCKAY CONJECTURE

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1. The main conjecture

Let Λ be the **ring of symmetric functions** in a countable set of variables $x_1, x_2, \ldots$ over $\mathbb{Q}$. Its most important basis are the Schur functions. There is one Schur function s_λ for each partition λ, and these functions form an orthonormal basis of Λ with respect to a distinguished inner-product $\langle\, ,\, \rangle$. An element of Λ is said to be **Schur-positive** if it is a non-negative integer combination of the Schur functions s_λ.

A nice way of thinking about s_λ is as a (formal) sum over standard tableau of shape λ:

$$s_\lambda = \sum_{\substack{T\text{standard}\\ \lambda-\text{tableau}}} m^T.$$

Via the Frobenius-isometry, s_λ corresponds to an irreducible character of the symmetric group S_n, where $n = |\lambda|$. Then the inner-product is induced by the usual inner-product of characters of the finite symmetric groups $S_1, S_2, \ldots$ (characters of different symmetric groups are orthogonal to each other).

We identify partitions with their Young diagrams, oriented in matrix (English) fashion. Thus each partition is represented by a set of boxes in the plane.

Whenever we have a partition α contained in a partition λ, we call the boxes in λ which are not in α a skew-diagram, denoted λ/α.

Multiplication of schur functions represents taking the outer tensor product of characters in a Young subgroup and inducing to the parent symmetric group. Then

$$s_\alpha s_\beta = \sum_\lambda C^\lambda_{\alpha,\beta} s_\lambda,$$

where the **Littlewood-Richardson** coefficient $C^\lambda_{\alpha,\beta}$ is a non-negative integer; the number of L-R tableau of shape λ/α and weight β.

We define the skew-schur symmetric function

$$s_{\lambda/\alpha} := \sum_\beta C^\lambda_{\alpha,\beta} s_\beta.$$

So $s_{\lambda/\alpha}$ is Schur positive, by its definition. Setting $\gamma := \lambda/\alpha$, we also use the notation s_γ for $s_{\lambda/\alpha}$ and $C_{\gamma,\beta}$ for $C^\lambda_{\alpha,\beta}$. A pretty fact is that

$$s_\gamma = \sum_{\substack{T\text{standard} \\ \gamma-\text{tableau}}} m^T.$$

In particular s_γ depends only on the shape of γ.

Our conjecture makes us of the **triangular partitions**:

$$\delta_t = (t, t-1, \dots, 3, 2, 1), \quad t \geq 0.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. A removable box of λ is a box in the diagram λ which can be removed to leave a partition diagram. Similarly an addable box of λ is a box in the plane which does not belong to the diagram of λ but whose addition to that diagram produces a partition diagram.

Let α be any subset of addable nodes of λ, and set $t := |\alpha|$. We denote by $\lambda \cup \alpha$ the partition obtained by adding all these boxes to λ. Now $\delta_t \subseteq \lambda$ and $\delta_{t+1} \subseteq \lambda \cup \alpha$. So we can form skew-partitions λ/δ_t and $\lambda \cup \alpha/\delta_{t+1}$, and these have the same size.

Conjecture 1. *Let α be a set of addable nodes of λ. Setting $t = |\alpha|$, it holds*

$$s_{\lambda\cup\alpha/\delta_{t+1}} - s_{\lambda/\delta_t} \quad \textit{is Schur positive.}$$

Example: let α be an addable box of λ. Then it is easy to show that $s_{\lambda\cup\alpha/\delta_1} = \sum_\mu s_\mu$, where μ ranges over the partitions of $|\alpha| - 1$ contained in α.

Example: take $t = 3$ addable boxes of $\lambda = (5, 4, 3, 1)$ to form $\lambda \cup \alpha = (6, 4, 4, 2, 1)$. Then SageMath gives:

$$s[5,4,3,1]/[3,2,1] =$$
$$s[2,2,2,1] + 2s[3,2,1,1] + 3s[3,2,2] + 3s[3,3,1]$$
$$+s[4,1,1,1] + 4s[4,2,1] + 2s[4,3] + s[5,1,1] + s[5,2]$$

$$s[6,4,4,2,1]/[4,3,2,1] =$$
$$s[2,2,1,1,1] + 2s[2,2,2,1] + s[3,1,1,1,1]+$$
$$5s[3,2,1,1] + 4s[3,2,2] + 4s[3,3,1] + 3s[4,1,1,1]$$
$$+7s[4,2,1] + 3s[4,3] + 3s[5,1,1] + 3s[5,2] + s[6,1].$$

Thus we see that $s[6,4,4,2,1]/[4,3,2,1] - s[5,4,3,1]/[3,2,1]$ is Schur positive.

Example: in the opposite direction, each of the following are Schur positive:

$$s[5,4,4,2,1]/[3,2,1] - s[5,4,3,1]/[2,1],$$
$$s[5,4,4,2,1]/[3,2,1] - s[4,4,3,2]/[2,1],$$
$$s[5,4,4,2,1]/[3,2,1] - s[4,4,4,1]/[2,1],$$
$$s[5,4,4,2,1]/[3,2,1] - s[4,4,3,1,1]/[2,1].$$

Commutativity of L-R coefficients enables us to rephrase our conjecture in terms of L-R fillings of triangular weight:

Conjecture 2 (Strong triangular L-R filling form)**.** *For a skew-partition λ/μ and a set of t addable nodes α of λ/μ:*

$$C_{(\lambda/\mu)\cup\alpha,\delta_{t+1}} \geq C_{\lambda/\mu,\delta_t}.$$

Example: $t = 3$, $\gamma = (5,4,3,1)/(3,2,1,1)$ and $\gamma \cup \alpha = (6,4,4,2,1)/(3,2,1,1)$. Then

$$C_{\gamma\cup\alpha,\delta_4} = 5 \quad > \quad 2 = C_{\gamma,\delta_3}.$$

This means that for a fixed t, we must only check a finite number of skew-partitions in order to verify the original conjecture for all skew-partitions λ/δ_t.

2. Odd characters of symmetric groups

We say that λ is an **odd partition** of n if the corresponding irreducible character of S_n has odd degree. A hook partition of an integer n is a partition of the form $(n-\ell, 1^\ell)$, where $0 \leq \ell < n$. We call ℓ the leg-length of the hook. More generally an **n-hook** is a connected skew-diagram with n boxes which contains no 2×2 subdiagram. The leg-length of an n-hook is one less than the number of its rows. When n is a power of 2, the hook-partitions are the odd partitions of n.

It is clear that there are 2^{n-1} different n-hooks, of which $\binom{n-1}{\ell}$ have leg-length ℓ (from a fixed point in the plane, take $n-1$ unit steps , each step either due east or due north).

Recall that a partition is said to be a **p-core** if it has no p-hooks, for $p > 0$. C. Bessenrodt noted that each p-core partition has exactly p addable p-hooks, distinguished by their leg-lengths.

Let $w \geq 0$ and suppose that $2w$ has binary decomposition:

$$2w = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_u}$$

with $b_1 > b_2 > \cdots > b_u > 0$. Let σ be a permutation of $2w$ of cycle type $(2^{b_1}, 2^{b_2}, \ldots, 2^{b_u})$, let λ be a partition of $2w$ and let χ be the corresponding irreducible character of S_{2w}. By Murnaghan-Nakayama $\chi(\sigma) \neq 0$ iff we can successively strip a 2^{b_i}-hook off λ, for $i = 1, \ldots, u$. Moreover, if $\chi(\sigma) \neq 0$, its value is ± 1. But $\chi(\sigma) \equiv \chi(1) \pmod 2$. Thus we get a criterion for λ to be odd in terms of its hook-lengths.

We may reverse the process of stripping hooks. Starting with the empty partition, we may successively add a 2^{b_i}-hook for $i = u, u-1, \dots, 1$ to obtain an odd partition of $2w$. Taking into account the 2^{b_i} choices for the leg-length of a 2^{b_i}-hook, this implies that

there are $2^{b_1+b_2+\cdots+b_u}$ odd partitions of $2w$.

Moreover, we see that the odd partitions are parametrized by u-tuples $\overline{h} = (h_1, \dots, h_u)$ of 2^{b_i}-hooks, for $i = 1, \dots, u$. Here h_i denotes a 2^{b_i}-hook, with a specified leg-length. We use the notation $\lambda(\overline{h})$ for an odd partition of $2w$.

Example: there are 8 odd partitions of 6, 64 odd partitions of 14, but only 16 odd partitions of 16.

3. Height zero characters in 2-blocks

The p-core of a partition λ is obtained by successively stripping p-hooks from λ. The partitions of n which share a p-core form a **p-block of partitions** of n. When p is prime, this describes the partition of the irreducible complex characters of S_n into Brauer p-blocks.

An odd partition of $2w$ has 2-core (), and an odd partition of $2w+1$ has 2-core (1) (and the corresponding 2-block is called the principal 2-block). More generally **2-cores are triangular partitions** $\delta_t := (t, t-1, \ldots, 2, 1)$, for $t \geq 0$.

Now consider the t-th triangular number $n_t := t(t+1)/2$, for $t \geq 0$. For each $w \geq 0$, $n_t + 2w$ has a 2-block of partitions with 2-core δ_t. We denote this block as $B(w,t)$, and we call w the 2-weight of $B(w,t)$.

J. Olsson (and others) showed that the number of irreducible characters (and modular characters) in $B(w,t)$ depends only on the weight w. M. Enguehard showed that there is one perfect isometry class of blocks of weight w, and J. Scopes showed that there are at most w Morita equivalence classes of 2-blocks of symmetric groups which have weight w (and we can find representatives of each class in S_N for some $N \leq w^2+1$). Her general method proves Donovan's conjecture for symmetric groups.

We say that a partition λ in $B(w,t)$ has **height zero** iff the 2-part of the degree of the corresponding irreducible character of $n_t + 2w$ coincides with the 2-part of n_t (this is the minimal power of 2 dividing the degree of any character in $B(w,t)$).

Now let $\sigma \in S_{n_t+2w}$ have cycle type $(2^{b_1}, 2^{b_2}, \ldots, 2^{b_u}, 1^{n_t})$. Again by Murnaghan-Nakayama, $\chi(\sigma) \neq 0$ iff we can successively strip a 2^{b_i}-hook off λ, for $i = 1, \ldots, u$. Moreover, if $\chi(\sigma) \neq 0$, its value is $\pm$ the degree of the irreducible character of S_{n_t} corresponding to the 2-core δ_t. Thus we get a criterion for λ to be of height zero (in $B(w,t)$) in terms of its hook-lengths.

Conversely, starting with the 2-core δ_t, we may successively add one 2^{b_i}-hook for $i = u, u-1, \ldots, 1$. This constructs a height zero partition of $n_t + 2w$. Thus

there are $2^{b_1+b_2+\cdots+b_u}$ height zero partitions in $B(w,t)$.

Moreover, we see that the height zero partitions are parametrized by u-tuples $\overline{h} := (h_1, \ldots, h_u)$ of 2^{b_i}-hooks, for $i = 1, \ldots, u$. Here h_i denotes a 2^{b_i}-hook, with specified leg-length. We use the notation $\lambda(\overline{h}, t)$ for a height zero partition in $B(w,t)$.

Summary: There is a (combinatorially defined) bijection $\lambda(\overline{h}, t) \leftrightarrow \lambda(\overline{h}, 0)$ between the height zero partitions in $B(w,t)$ and the odd partitions of $2w$.

4. Restriction to a Young subgroup

Consider the Young subgroup $S_{n_t} \times S_{2w}$ of S_{n_t+2w}. This has a 2-block of irreducible characters, formed by tensoring the character of δ_t with the characters of the partitions in the principal 2-block $B(w,0)$ of S_{2w}. This 2-block is the Brauer correspondent of $B(w,t)$ in $S_{n_t} \times S_{2w}$. We may induce each of these characters from $S_{n_t} \times S_{2w}$ to S_{n_t+2w}. The decomposition of the induced character into irreducible characters is governed by the Littlewood-Richardson rule:

$$(\delta_t \otimes \lambda) \uparrow^{S_{n_t+2w}} = \sum_{\alpha \vdash n_t+2w} C^{\alpha}_{\delta_t,\lambda} \alpha.$$

As above the *Littlewood-Richardson* coefficient $C_{\delta_t,\lambda}$ counts the number of L-R tableau of shape α/δ_t and weight λ.

Our conjecture on skew-Schur functions arose from asking whether the irreducible character of $S_{n_t} \times S_{2w}$ corresponding to $\delta_t \otimes \lambda(\overline{h})$ occurs in the restriction of the irreducible character of S_{n_t+2w} corresponding to $\lambda(\overline{h}, t)$. This leads to a weak form of our conjecture:

Conjecture 3 (2-local restriction)**.** *Given an odd partition $\lambda(\overline{h})$ of $2w$, and $t \geq 0$, we have $C^{\lambda(\overline{h},t)}_{\delta_t,\lambda(\overline{h})} > 0$.*

Example: let $w > 0$ and consider the 'staircase' $(w+1, w, w-1, \ldots, 2)/(w-1, w-2, \ldots, 2, 1)$.

It is a skew-partition with $2w$ boxes. Now the 'almost' triangular partition $(w+1, w, w-1, \ldots, 2)$ has height zero in $B(2w, w-1)$, and corresponds to the u-tuple $(h_1, \ldots, h_u)$ of hooks, where $h_i = (2^{b_i-1}+1, 1^{2^{b_i-1}-1})$ is the 2^{b_i}-hook with leg-length $2^{b_i-1}-1$, for $i = 1, \ldots, u$.

The following L-R coefficients $C_w := C^{\lambda(\overline{h}, w-1)}_{\delta_{w-1}, \lambda(\overline{h})}$ were computed in SageMath:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14
C_w	1	1	1	1	9	49	89	1	49	1737	16121	281521	10414313	177901001

The only thing I know about these L-R coefficients is that they are non-zero. However it seems that the coefficients are odd, and indeed are congruent to 1 mod 4.

More generally, in the conjext of the weak conjecture, there are numerous examples where $C^{\lambda(\overline{h}, t)}_{\delta_t, \lambda(\overline{h})}$ is even. So I do not claim oddness in general.

5. Growth diagrams of height zero partitions

C. Bessenrodt used bead sequences (also known as Comét codes) to prove results about the hook lengths of partitions.

A few years ago I used her formulation to prove the following result about p-cores. This may be of general interest (the proof is not too hard):

Lemma 4. *Suppose that $\alpha \supseteq \beta$ are p-cores, with addable p-hooks of the same leg-length h_α and h_β. Then*

$$\alpha \cup h_\alpha \supseteq \beta \cup h_\beta.$$

Next, recall that we have parametrized the height zero partitions in $B(w,t)$ by u-tuples of hooks. In particular there is a bijection between the the height zero partitions in $B(w,t)$ and those in $B(w,t+1)$, for all $t \geq 0$. Now 2-cores form an ascending chain of partitions:

$$() \subset (1) \subset (2,1) \subset (3,2,1) \subset (4,3,2,1) \subset \dots$$

We obtain the following consequence:

Theorem 5. *Let $w > 0$ and let $\lambda(\overline{h})$ be an odd partition of $2w$. Then the corresponding height zero partitions in $B(2w,t)$, for $t = 0,1,2,\dots$, form an ascending chain:*

$$\lambda(\overline{h}) \subset \lambda(\overline{h},1) \subset \lambda(\overline{h},2) \subset \lambda(\overline{h},3) \subset \dots$$

Moreover the $t+1$ boxes in $\lambda(\overline{h},t+1)/\lambda(\overline{h},t)$ have the same 2-residue t. In particular they are addable boxes of $\lambda(\overline{h},t)$.

Example: $w = 7$. So $2w = 8+4+2$. Choose $8, 4, 2$ hooks of leg-lengths $3, 1, 0$ respectively. Then the chain is:

$$(5, 4^2, 1) \subset (5, 4^2, 2) \subset (5^2, 4, 3) \subset (5^3, 4, 1) \\ \subset (6, 5^3, 2, 1) \subset (7, 6, 5^2, 3, 2, 1) \subset (8, 7, 6, 5, 4, 3, 2)$$

We can strengthen our weak conjecture to:

Conjecture 6 (Ascending)**.** *Given an odd partition* $\lambda(\overline{h})$ *of* $2w$*, and* $t \geq 0$*, we have* $C^{\lambda(\overline{h},t+1)}_{\delta_{t+1},\lambda(\overline{h})} \geq C^{\lambda(\overline{h},t)}_{\delta_t,\lambda(\overline{h})}$.

Example: for the chain of heigh-zero partitions above

t	1	2	3	4	5	6	7
$C^{\lambda(\overline{h},t)}_{\delta_t,\lambda(\overline{h})}$	1	1	1	1	3	29	89

Our observation on the chain of height-zero partitions implies the following interesting fact:

Lemma 7. *Given an odd partition λ, the complement of λ in the positive quadrant can be filled by i symbols i, for $i = 1, 2, \ldots,$ such that for each $t \geq 0$, the symbols $\leq t$, together with λ, form the diagram of an odd partition of $|\lambda| + n_t$.*

Now of course $C^{\lambda}_{\alpha,\beta} = C^{\lambda}_{\alpha,\beta}$, for all partitions α, β, λ. Recall that a **semi-standard tableau** is a filling of a skew-partition by numbers which are weakly increasing along rows and strictly increasing along columns.

We use the term **increasing tableau** if the numbers increase along both rows and columns. The weight of the tableau is $(m_1, m_2, \ldots)$ where m_i is the number of symbols i in the tableau. The above conjecture naturally leads to:

Conjecture 8 (Weak triangular L-R form)**.** *Let $t > 0$ and let γ be a skew-partition. Then C_{γ,δ_t} is not less than the number of increasing γ-tableau of weight $(1, 2, 3, \ldots, t)$.*

Example: consider $\delta = (6, 5^3, 2, 1)/(5, 4^2, 1)$. Then there are three L-R fillings of γ of weight $(4, 3, 2, 1)$ but only one increasing γ-tableau of weight $(1, 2, 3, 4)$.

This weak triangular L-R conjecture led directly to the strong triangular L-R conjecture, which in turn is equivalent to Conjecture 1.

6. Positive evidence from combinatorics

For a skew-partition γ, $\mathrm{rows}(\gamma)$ is the partition formed by ordering the lengths of the rows of γ and $\mathrm{cols}(\gamma)$ is the partition formed by ordering the lengths of the columns of γ. It is folklore that if δ is any skew-partition such that $s_\gamma - s_\delta$ is Schur positive, then

$$\mathrm{rows}(\gamma) \leq \mathrm{rows}(\delta), \quad \text{and} \quad \mathrm{cols}(\gamma) \leq \mathrm{cols}(\delta).$$

Here $\leq$ is the dominance order.

P. McNamara has sharpened these inequalities [J. Algebraic Comb. 28 (4) (2008)]. For all $k, \ell \geq 1$, $\mathrm{rect}_{k,\ell}(\gamma)$ is defined to be the number of $k \times \ell$ rectangles contained in γ. Then McNamara proved:

Theorem 9. *If $s_\gamma - s_\delta$ is Schur positive, then*

$$\mathrm{rect}_{k,\ell}(\gamma) \leq \mathrm{rect}_{k,\ell}(\delta).$$

However these inequalities are not sufficient to ensure Schur positivity, by any means. However we have managed to prove the following, in the context of Conjecture 1:

Theorem 10. *Let α be a set of addable nodes of λ and set $t = |\alpha|$. Then*

$$\mathrm{rect}_{k,\ell}(\lambda \cup \alpha/\delta_{t+1}) \leq \mathrm{rect}_{k,\ell}(\lambda/\delta_t), \quad \textit{for all } k, \ell \geq 1.$$