Cores of Ariki-Koike algebras

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Representation Theory of Symmetric groups

Over a field of characteristic zero:

$$\mathrm{Irr}(\mathfrak{S}_n) \leftrightarrow \Pi(n) = \text{Partitions of } n$$

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Over a field of characteristic zero:

$$\mathrm{Irr}(\mathfrak{S}_n) \leftrightarrow \Pi(n) = \text{Partitions of } n$$

- We have a set of Specht modules S^λ (defined over $\mathbb{Z}$) with $\lambda \in \Pi(n)$.
- For all field $\mathbb{F}$, $\mathbb{F}S^\lambda := \mathbb{F} \otimes_{\mathbb{Z}} S^\lambda$ is a $\mathbb{F}\mathfrak{S}_n$-module.
- If $\mathrm{car}(k) = 0$,

$$\mathrm{Irr}(\mathbb{F}\mathfrak{S}_n) = \left\{\mathbb{F}S^\lambda \mid \lambda \in \Pi(n)\right\}$$

In positive charcteristic ? for all λ, we have composition series:

$$0 \subset M_1 \subset \ldots \subset \mathbb{F}S^\lambda$$

and well-defined multiplicities $[\mathbb{F}S^\lambda : M]$ for all $M \in \mathrm{Irr}(\mathbb{F}\mathfrak{S}_n)$

Problem

Find the decomposition matrix:

$$D^{\mathbb{F}} = ([\mathbb{F}S^\lambda : M])_{\lambda \in \Pi(n), M \in \mathrm{Irr}(\mathbb{F}\mathfrak{S}_n)}$$

$\rightsquigarrow$ Difficult ! even the dimension of the simples are not known in general ...

Labelling of the simples:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots & & \vdots \\ * & & & 1 & & \vdots \\ * & & & & & 1 \\ * & & & & & * \\ \vdots & & & & & \vdots \end{bmatrix}.$$

The first rows are labeled by the set of p-regular partitions.
No part are repeated p or more times.

Example: $p = 3$, $n = 5$

$$\begin{array}{c} (5) \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1.1) \\ (1.1.1.1.1) \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Hecke algebras

Let $q \in \mathbb{F}^\times$ with a field $\mathbb{F}$.

Hecke algebra $\mathcal{H}(q, \mathbb{F})$ of the symmetric group:

- Basis: $\{T_w \mid w \in \mathfrak{S}_n\}$
- Relations: $T_{s_i}^2 = qT_1 + (q-1)T_{s_i}$ for $s_i = (i, i+1)$
  $T_w = T_{s_{i_1}} \dots T_{s_{i_t}}$ pour $w = s_{i_1} \dots s_{i_t}$ with t minimal.

q-Deformation of $\mathbb{F}\mathfrak{S}_n$. We have quantum analogues:

- of the Specht modules: $\mathbb{F}S_q^\lambda$,
- of the decomposition matrix

$$D_q^{\mathbb{F}} = ([\mathbb{F}S_q^\lambda : M])_{\lambda \in \Pi(n), M \in \mathrm{Irr}(\mathcal{H}(q,\mathbb{F}))}$$

If $q=1 \to \mathcal{H}(q,\mathbb{F}) = \mathbb{F}\mathfrak{S}_n$

In general, $D^{\mathbb{C}}_{\eta_e}$ is a "good" approximation of $D^{\mathbb{F}_e}$

Set

$$e = \min\{i \geqslant 2 \mid 1+q+q^2+\ldots+q^{i-1}=0\}$$

$\to$ The structure of the blocks only depends on e.

Block Theory

Let $e \in \mathbb{N}_{>1}$, for any field.

- Two partitions are in the same block if they are in the same block of the decomposition matrix.
- In other words: let $(\lambda, \mu) \in \Pi(n)^2$ then set

$$\lambda \equiv \mu$$

if there exists $M \in \mathrm{Irr}(\mathcal{H}(q, \mathbb{F}))$ such that

$$[\mathbb{F}S_q^\lambda : M] \neq 0 \text{ and } [\mathbb{F}S_q^\mu : M] \neq 0$$

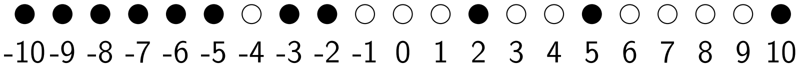
Being in the same block
$\iff$ in the same equivalence class for the transitive closure of the relation $\equiv$.

Example: $p=3$, $n=5$

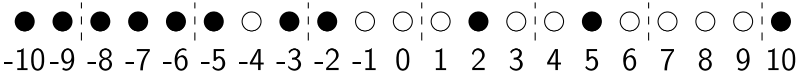
$$\begin{array}{c} (5) \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1.1) \\ (1.1.1.1.1) \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

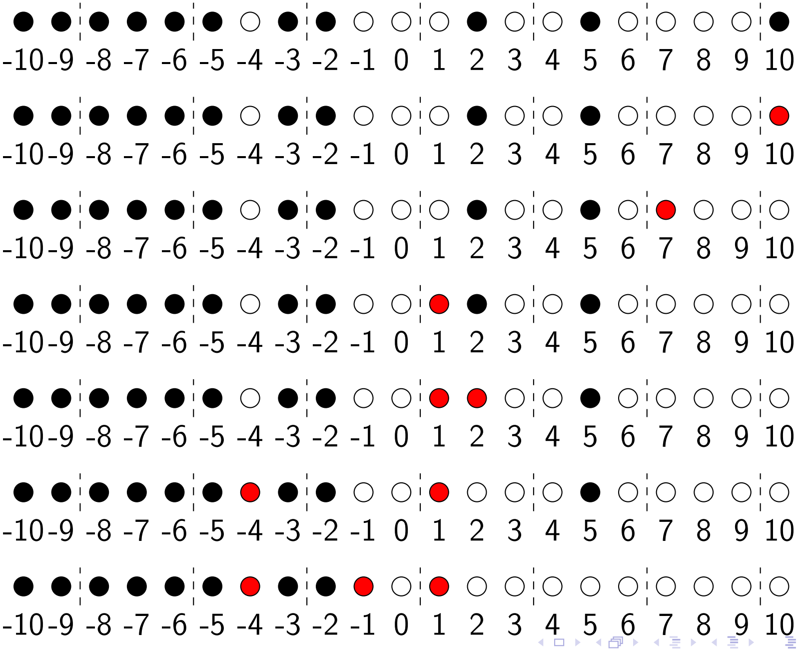
Nakayama conjecture

- Two partitions are in the same block if and only if they have the same e-core.
- The e-core of a partition can be computed by removing the e-hooks of the partition.
- It can also be computed using the abacus decomposition.
- Take the abacus of the partition, for example $\lambda = (10, 6, 4, 1, 1)$



- Take $e = 3$ then:




-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
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The 3-core of the partition $(10, 6, 4, 1, 1)$ is the partition (1).

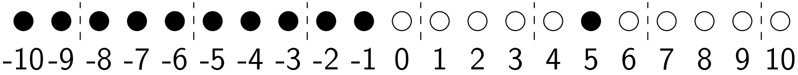
- The number of moves of beads in the abacus to get an e-core is the e-weight of the partition.
- The weight measures the complexity of the block.
- Blocks of e-weight 0 are singleton given by e-cores
- Blocks of e-weight 1 are very easy to describe,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- The decomposition numbers of the blocks of e-weight 2 and 3 are known (Fayers, Lyle-Ruff).

Example: $p = 3$, $n = 5$

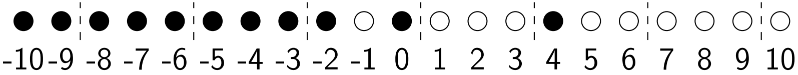
$$\begin{array}{c} \color{red}{(5)} \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1,1) \\ (1.1.1.1.1) \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$w(5) = 1$$

Example: $p = 3$, $n = 5$

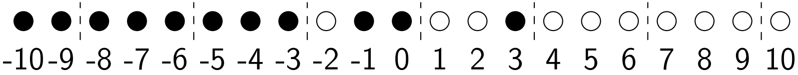
$$\begin{array}{c} (5) \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1,1) \\ (1.1.1.1.1) \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$w(4.1) = 1$$

Example: $p = 3$, $n = 5$

$$\begin{array}{c} (5) \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1,1) \\ (1.1.1.1.1) \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$w(3.1.1) = 0$$

Example: $p = 3$, $n = 5$

$$\begin{matrix} (5) \\ (4.1) \\ (3.2) \\ \color{red}{(3.1.1)} \\ (2.2.1) \\ (2.1.1, 1) \\ (1.1.1.1.1) \end{matrix} \quad \begin{pmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

Example: $p = 3$, $n = 5$

$$\begin{array}{c} (5) \\ (4.1) \\ (3.2) \\ \color{red}{(3.1.1)} \\ (2.2.1) \\ (2.1.1,1) \\ (1.1.1.1.1) \end{array} \begin{pmatrix} . & . & . & 0 & . \\ . & . & . & 0 & . \\ . & . & . & 0 & . \\ 0 & 0 & 0 & 1 & 0 \\ . & . & . & 0 & . \\ . & . & . & 0 & . \\ . & . & . & 0 & . \end{pmatrix}$$

$(3.1.1)$ is a 3-core

Example: $p = 3$, $n = 5$

$$\begin{matrix} (5) \\ (4.1) \\ (3.2) \\ (3.1.1) \\ (2.2.1) \\ (2.1.1.1) \\ (1.1.1.1.1) \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & 0 & . \\ 0 & . & 0 & 0 & . \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & . \\ . & . & 1 & 0 & . \\ . & . & 0 & 0 & . \end{pmatrix}$$

(5), (2.2.1) and (2.1.1.1) have (2) as a 3-core (weight 1)

Example: $p = 3$, $n = 5$

$$\begin{array}{c} (5) \\ \color{red}{(4.1)} \\ \color{red}{(3.2)} \\ (3.1.1) \\ (2.2.1) \\ (2.1.1.1) \\ \color{red}{(1.1.1.1.1)} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(4.1), (3.2) and (1.1.1.1.1) have (1.1) as a 3-core (weight 1)

Ariki-Koike algebras

Let $\mathbb{F}$ a field and $q \in \mathbb{F}^{\times}$. Let $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbb{Z}^l$. The Ariki-Koike algebra $\mathcal{H}^{\mathbf{s}}(q, \mathbb{F})$ is presented by

- generators by $T_0, T_1, \ldots, T_{n-1}$
- Relations:
  - $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$
  - $T_i T_j = T_j T_i$ if $|i - j| > 1$
  - $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
  - $(T_i - q)(T_i + 1) = 0$ if $i > 0$
  - $(T_0 - q^{s_1}) \ldots (T_0 - q^{s_l}) = 0.$

It can be seen as a generalization of the Hecke algebra.

- Includes Hecke algebras of type A ($l = 1$) and B ($l = 2$)
- Quotient of the affine Hecke algebra of type A
- Connected with Cherednik algebras, Calogero-Moser cells ...
- Representations of reductive groups, Spetses etc.
- Quantum affine algebras.
- Quiver-Hecke algebras

Representation Theory

If $\mathcal{H}^s(q,\mathbb{F})$ is semisimple:

$$\mathrm{Irr}(\mathcal{H}^s(q,\mathbb{F})) \leftrightarrow \Pi^l(n)$$

where

$$\Pi^l_n = \{(\lambda^1,\lambda^2,\dots,\lambda^l) \mid \forall i \lambda^i \in \Pi(n_i),\ \ n_1+\ldots+n_l = n\}$$

In general:

- We have a set of $\mathcal{H}^s(q,\mathbb{F})$-module modules $\mathbb{F}S_q^\lambda$ with $\lambda \in \Pi^l(n)$ ("Specht modules")
- In the semisimple case

$$\mathrm{Irr}(\mathcal{H}^s(q,\mathbb{F})) = \left\{\mathbb{F}S_q^\lambda \mid \lambda \in \Pi^l(n)\right\}$$

In the non semisimple case,

$$e = \min\{i \geqslant 2 \mid 1 + q + q^2 + \ldots + q^{i-1} = 0\}$$

- One can assume that $\mathbf{s}$ in the set

$$\mathcal{A}_e^l := \{(s_1, \ldots, s_l) \in \mathbb{Z}^l \mid \forall 1 \leqslant i < j \leqslant l,\ 0 \leqslant s_j - s_i < e\}.$$

- we have a decomposition matrix:

$$D^{\mathbb{F}}_{\mathbf{s},q} = ([\mathbb{F}S_q^\lambda : M])_{\lambda \in \Pi^l(n), M \in \mathrm{Irr}(\mathcal{H}^{\mathbf{s}}(q,\mathbb{F}))}$$

- If $\mathbb{F} = \mathbb{C}$, explicit combinatorial algorithm for computing it (Ariki, Lascoux-Leclerc-Thibon, Uglov, J)
- $D^{\mathbb{F}}_{\mathbf{s},q} \rightsquigarrow$ Generalization of p-regular partitions (Kleshchev, FLOTW, Uglov l-partitions)
- What can we say about blocks ?

Weight and blocks

- $\lambda = (\lambda^1, \ldots, \lambda^l) \in \Pi^l(n)$,
- Residues of λ: for $l = 2$, $\mathbf{s} = (0, 1)$ and $e = 3$ the residues of the nodes of the 2-partition $((4), (2, 1))$ of 7 are given as follows:

$$\left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 0 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$$

- $c_i(\lambda)$ is the number of nodes with residue i in the l-partition.
- $\mathcal{C}_{e,\mathbf{s}}(\lambda) := (c_0(\lambda), \ldots, c_{e-1}(\lambda))$.
- Here we have $\mathcal{C}_{e,\mathbf{s}}((4), (2, 1)) = (3, 2, 2)$.

- (Lyle-Mathas) Two l-partitions λ and μ are in the same block if and only if

$$\mathcal{C}_{e,\mathbf{s}}(\lambda) = \mathcal{C}_{e,\mathbf{s}}(\mu)$$

- For example for $l = 2$, $\mathbf{s} = (0,1)$ and $e = 3$, the partition $((4),(2,1))$ is in the same block as

$$\left(\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 0 \\ \hline 0 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right)$$

- Fayers: Notion of weight:

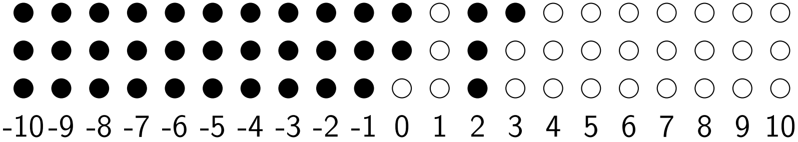
$$p_{(e,\mathbf{s})}(\lambda) = \sum_{1 \leqslant i \leqslant l} c_{s_i}(\lambda) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i(\lambda) - c_{i-1}(\lambda))^2.$$

- Can we find a similar characterization as in the case of the symmetric group ?

Let $s \in \mathbb{Z}^l$. and $\lambda \in \Pi^l(n)$. Write

$$(L_{s_1}, \ldots, L_{s_l})$$

its abacus (e.g. $s = (0, 1, 2)$ and $\lambda = ((2), (1), (1.1)))$

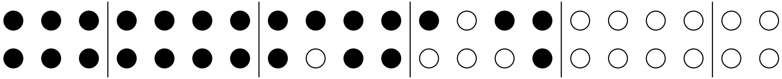


then we say that λ is a *reduced* (e, s)*-core*

1. $l = 1$ and $L_{s_1}(\lambda^1) \subset L_{s_1+e}(\lambda^1)$,
2. or $l > 1$ and

$$L_{s_1}(\lambda^1) \subset L_{s_2}(\lambda^2) \subset \ldots \subset L_{s_l}(\lambda^l) \subset L_{s_1+e}(\lambda^1).$$

For $s = (0, 3)$ and $e = 4$, $\lambda = ((4, 1, 1), (1, 1))$ is a reduced (e, s)-core.



Proposition (J-Lecouvey, Fayers)

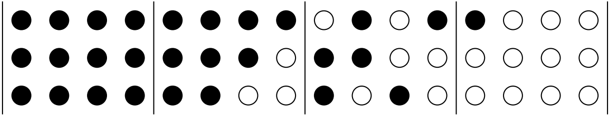
The reduced (e, s)-cores are exactly the multipartitions with e-weight 0.

Is is then easy to see that the associated blocks are simple.

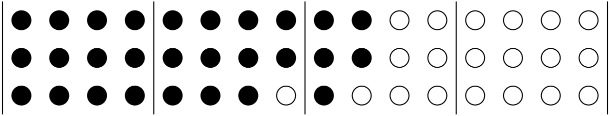
Theorem (J-Lecouvey)

Two l-partitions with the same rank have the same core if and only if they belong to the same block.

Computation of the weight

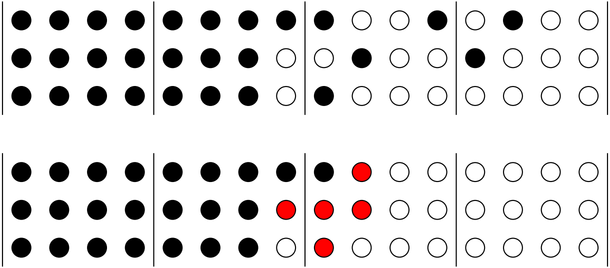


To determine its core, we perform the above procedure and we obtain the following 3-abacus:



the associated $(e, \mathbf{s})$-core is the 3-partition $((1), \emptyset, \emptyset)$ together with the multicharge $(0, 2, 2)$

Example



Take $n = 4$, $e = 4$ and $\mathbf{s} = (0, 1)$.

2-partition	core	block weight
$((4), \emptyset)$	$(\emptyset; (0, 1))$	2
$((3), (1))$	$((\emptyset, (1, 1)); (0, 3))$	1
$(\emptyset, 4)$	$(\emptyset; (0, 1))$	2
$((3, 1), \emptyset)$	$(\emptyset; (0, 1))$	2
$((2), (2))$	$((\emptyset, 1.1); (0, 3))$	1
$((1), (3))$	$(\emptyset; (0, 1))$	2
$((2, 2), \emptyset)$	$(((2), \emptyset); (0, 3))$	1
$((2, 1), (1))$	$(((2, 1), 1); (0, 1))$	0
$((2, 1, 1), \emptyset)$	$(\emptyset; (0, 1))$	2
$((2), (1, 1))$	$(((2), (1, 1)); (0, 1))$	0

2-partition	core	block weight
$((1,1),(2))$	$(\emptyset;(0,1))$	2
$((1),(2,1))$	$(((1),(2,1));(0,1))$	0
$((1,1),(1,1))$	$(((2),\emptyset);(0,3))$	1
$(\emptyset,(3,1))$	$(\emptyset;(0,1))$	2
$((1,1,1),(1))$	$(\emptyset;(0,1))$	2
$(\emptyset,(2,2))$	$((\emptyset,(1,1));(0,3))$	1
$((1,1,1,1),\emptyset)$	$(\emptyset;(0,1))$	2
$(\emptyset,(2,1,1))$	$(\emptyset;(0,1))$	2
$((1),(1,1,1))$	$(((2),\emptyset);(0,3))$	1
$(\emptyset,(1,1,1,1))$	$(\emptyset;(0,1))$	2

- Corresponds to Fayers definition of weight.
- Generalization of the notion of weights.
- Blocks of weight 1 and 2 are known (Fayers, J-Lecouvey)
- The multicharge associated to the l-partition may be different from the one associated with its core !!

Schur elements

- Ariki-Koike algebra = Hecke algebra of the complex reflection group $G(l,1,n)$
- We have a one parameter Hecke algebra H of type $G(l,p,n)$
- In the semisimple case, each simple H-module $\rightsquigarrow$ Schur element $s_V(q) \in \mathbb{C}[q,q^{-1}]$.
- H is semisimple unless the parameter is a e-root of 1.

The *e-defect* (e-weight) of V is denoted by $d_e(V)$ and it is the maximal element $k \in \mathbb{N}_{\geqslant 0}$ such $\Phi_e(q)^k$ divides the Schur element $s_V(q)$.

Theorem (Chlouveraki-J)

Under the above hypotheses, assume that $W = G(l, p, n)$*, if two simples modules are in the same block then they have the same e-defect.*