# On self-extensions of irreducible modules for $\mathfrak{S}_n$

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Joint work with S. Kleshchev and L. Morotti

OIST, 11 October 2022

# The conjecture

**Conjecture (Kleshchev-Martin, 198-)**

*Let $\Bbbk$ be a field of odd characteristic. For an irreducible $\Bbbk\mathfrak{S}_n$-module $D$ we have*

$$\mathrm{Ext}^1_{\Bbbk\mathfrak{S}_n}(D, D) = 0.$$

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$$\Bbbk\mathfrak{S}_2 \cong \frac{\Bbbk}{\Bbbk}$$

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- (Kleshchev-Nakano, 01) If $p \geq 5$ and $D^\lambda = \mathrm{Sp}(\mu)$ for some $\mu \in \mathscr{P}(n)$ then

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$$\mathrm{Sp}(4, 1^7) = D^{(4,2,2,2,1)} \checkmark$$

# Blocks

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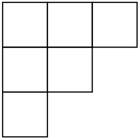
$[\lambda] =$ 

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- For $V \in \Bbbk\mathfrak{S}_n\text{-}\mathrm{mod}$,

$$V = \bigoplus_{\theta \in \Theta_n} V[\theta]$$

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$$\mathrm{Ext}^k_{\mathfrak{S}_{n-r}}(e_i^{(r)}V, W) = \mathrm{Ext}^k_{\mathfrak{S}_n}(V, f_i^{(r)}W) \quad \text{and}$$
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$$V_i / V_{i-1} \cong \mathrm{Sp}(\lambda^i)$$

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$$0 \to \mathrm{rad}(f_i^{(r)} D^\mu) \to f_i^{(r)} D^\mu \to D^\lambda \to 0$$

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Apply $\mathrm{Hom}_{\mathfrak{S}_n}(-, D^\lambda)$,

$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{rad}(f_i^{(r)} D^\mu), D^\lambda) \to \mathrm{Ext}^1_{\mathfrak{S}_n}(D^\lambda, D^\lambda) \to \mathrm{Ext}^1_{\mathfrak{S}_n}(f_i^{(r)} D^\mu, D^\lambda)$$

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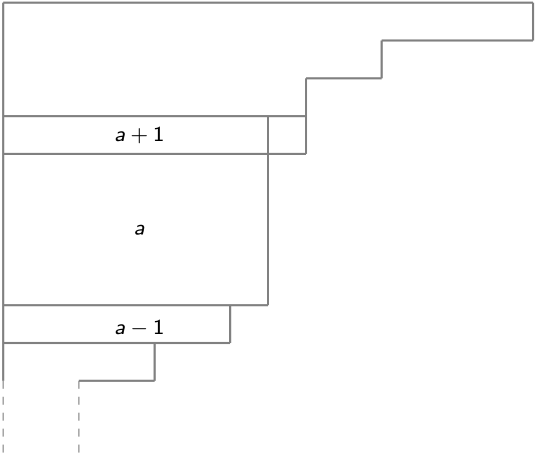
- **Criterion:**

$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{rad}(f_i^{(r)} D^\mu), D^\lambda) = 0,$$

unless $\lambda = (\lambda_1, \ldots, \lambda_k, a+1, a^{p-2}, a-1, \lambda_{k+p+1}, \ldots, \lambda_n)$

# Results

$[\lambda] =$ 

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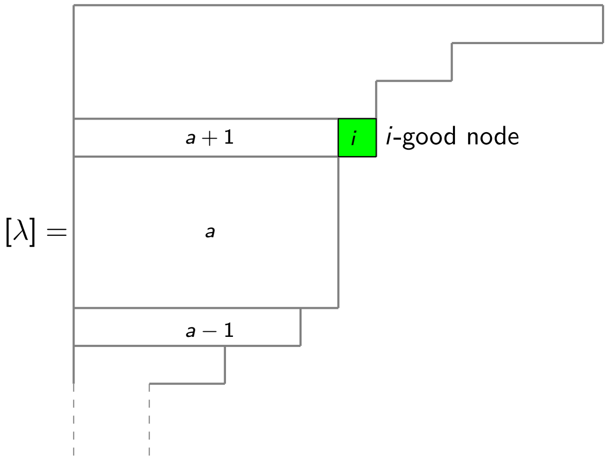

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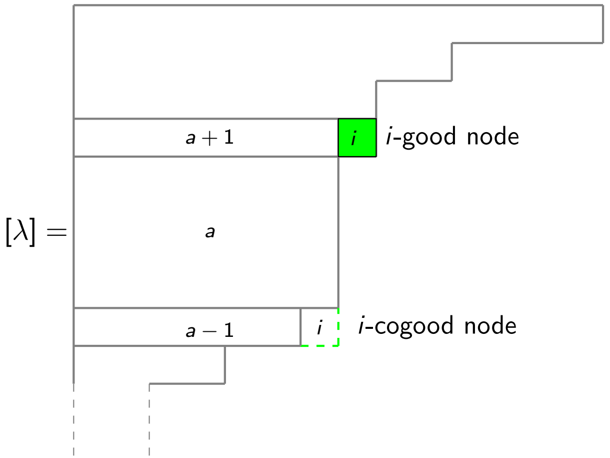

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$$D^{(2,1^{p-2})} = Sp(1^p) \checkmark$$

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$$\begin{aligned} e_0^{(2)} D^\lambda &= D^{(p^2,(p+1)^{p-1},p,1^{p-1})} \\ &= \mathrm{Sp}(p^2, p+1, 2^{p-1}, 1^{p(p-1)-1}) \quad \checkmark \end{aligned}$$

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$$\mathrm{Hom}_{\mathfrak{S}_n}(X, D^\lambda) = 0 \Rightarrow \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{rad}(f_i^{(r)} D^\mu), D^\lambda) = 0$$

Thank you