Bruhat order and Verma modules

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Bruhat order

Let (W, S) be a *Coxeter group*. That is, the group W is generated by a finite set S subject to relations of the following form.

- the braid relation

$$\underbrace{st\cdots}_{m} = \underbrace{ts\cdots}_{m}, \text{ for each } s,t \in S \text{ with } m = m_{st} = m_{ts};$$

- the reflection relation

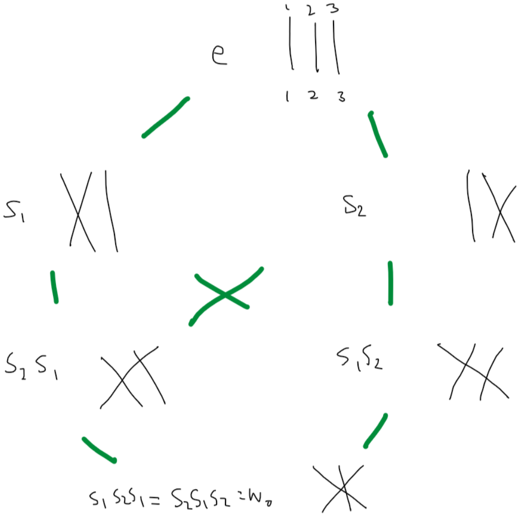
$$ss = e \text{ for each } s \in S;$$

Example. A symmetric group $(S_{n+1}, \{(1,2),(2,3),\cdots,(n,n+1)\})$

A *reduced expression* of $w \in W$ is a shortest expression of $w \in W$ as a product of elements in S.

The *Bruhat order* sets $w \leqslant x$ if a reduced expression of x has a (not nec. consecutive) subexpression which expresses w.

Example : $(S_3, \{s_1, s_2\})$



Join (and meet)

Let $(P, \leqslant)$ be a p(artially)o(rdered)set. The *join*, or suprimum, of a subset $U \subset P$ is the minimal element in $P_{\geqslant U} = \{a \in P \mid a \geqslant q \text{ for } q \in U\}$, if unique, and does not exist otherwise.

$$\vee : 2^P \to P \sqcup \{\text{“does not exist”}\}$$

Example. For $(S_3, \{s_1, s_2\})$,

- $s_1 \vee s_2$ does not exist;
- $s_1 \vee s_2 \vee s_1 s_2 = s_1 s_2$;
- $s_1 s_2 \vee s_2 s_1 = w_0$;

  etc.

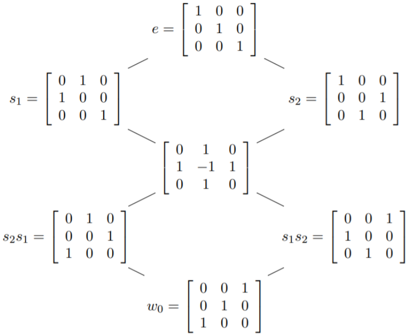
The *meet* (infimum) is the opposite.

Lattice completion

A *lattice* is a poset where all joins and meets exist.

(Dedekind-)MacNeille completion: Given a poset P, there is (a construction of) the smallest lattice that contains P. Denote it by $\mathrm{Lat}(P)$.

Example: For $P = (S_n, \leqslant)$, we have $\mathrm{Lat}(P) = \mathrm{ASM}_n$ the poset of $n \times n$ *alternating sign matrices* (identify S_n with permutation matrices).



Verma modules

Let $\mathfrak{g}$ be a finite semisimple complex Lie algebra, e.g., $\mathfrak{sl}_n$.

Let us fix Borel and Cartan subalgebras

$$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}.$$

The associated Weyl group W acts on $\mathfrak{h}^*$, the weights. E.g., S_n acts by permuting the diagonal entries.

The *Verma module* of highest weight $\lambda \in \mathfrak{h}^*$ is defined as

$$\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

These are the standard objects in the BGG category $\mathcal{O}$.

In particular

$$\operatorname{hd} \Delta(\lambda) \cong L(\lambda)$$

is the simple $\mathfrak{g}$-module of highest weight λ.

This classifies the simple modules in $\mathcal{O}$.

The poset of Verma modules

We restrict ourselves to the Verma modules whose highest weights are

$$w.0 = w(0 + \rho) - \rho$$

for $w \in W$, where 2ρ is the sum of all positive roots.

These are the standard objects in the principal block of category $\mathcal{O}$.

Our shorthand.

$$\Delta_w := \Delta(w.0) \quad L_w := L(w.0).$$

Setting $S = \{$ simple reflections in $W\}$, the Weyl group becomes a Coxeter group (W, S).

Fact. Each Δ_w is canonically a submodule of Δ_e via a unique (up to scalar) map $\Delta_w \hookrightarrow \Delta_e$.

We get the subposet

$$(\{\Delta_w\}_{w \in W}, \subset) \subset (\{\text{ submodules of } \Delta_e\}, \subset).$$

The Bruhat order and Verma modules

Fact. We have

$$w \geqslant x \iff \Delta_w \subset \Delta_x,$$

where $\leqslant$ is the Bruhat order. In other words,

$$\Delta : (W, \leqslant) \xrightarrow{\sim} (\{\Delta_w \mid w \in W\}, \supset).$$

Question 1. What is a representation theoretic characterization of $\mathrm{Lat}(W)$? I.e., what do we put in '???' so that the diagram commutes?

$$\begin{array}{ccc} (W, \leqslant) & \xrightarrow{\simeq} & (\{\Delta_w\}, \supset) \\ \cap & & \cap \\ \mathrm{Lat}(W) & \xrightarrow{\simeq} & \{M \subset \Delta_e \mid ???\} \end{array}$$

Note that $(\{M \subset \Delta_e\}, \supset)$ is a lattice (where the join is given by $\cap$ and the meet by the sum), thus contains $\mathrm{Lat}(\{\Delta_w\})$.

Intersections of Verma modules

Question was

$$\begin{array}{ccc} (W,\leqslant) & \xrightarrow{\simeq} & (\{\Delta_w\},\supset) =: \mathrm{Ver} \\ \cap & & \cap \\ \mathrm{Lat}(W) & \xrightarrow{??} & ??? \end{array}$$

Note any element in $\mathrm{Lat}(W)$ is written as the join of a subset in W.

Thus we get

$$\begin{array}{ccc} (W,\leqslant) & \xrightarrow{\simeq} & \mathrm{Ver} \\ \cap & & \cap \\ \mathrm{Lat}(W) & \xrightarrow{\phi} & \{M \subset \Delta_e \mid M = \bigcap_{w\in U} \Delta_w \text{ for some } U \subset W\} \end{array}$$

given by $\phi : \bigvee U \longmapsto \bigcap_{w\in U} \Delta_w$.

Question 1 and intersections of Verma modules

Consider the poset diagram

$$\begin{array}{ccc} (W, \leqslant) & \xrightarrow{\simeq} & (\{\Delta_w\}, \supset) = \mathrm{Ver} \\ \cap & & \cap \\ \mathrm{Lat}(W) & \xrightarrow{\phi} & \{M \subset \Delta_e \mid M = \bigcap_{w \in U} \Delta_w \text{ for some } U \subset W\} \end{array}$$

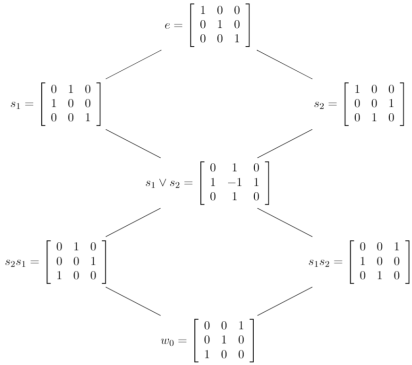
where $\phi : \bigvee U \longmapsto \bigcap_{w \in U} \Delta_w$.

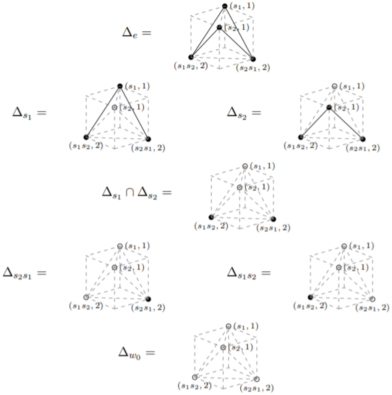
Theorem (K. 2021 and K.-Mazorchuk-Mrden 2021)

If $\mathfrak{g} = \mathfrak{sl}_n$, then the diagram commutes and ϕ is an isomophism.
In general, the diagram may not commute and ϕ may not be an isomorphism.

Note the meet $\wedge$ does not agree with the sum of modules!

Lat(S_3) vs $\{\bigcap_{w\in U}\Delta_w \mid U\subset S_3\}$





Tetrahedrons: to be explained

Weaker questions

Question 1w Does the poset diagram

$$\begin{array}{ccc} 2^W & \xrightarrow{\vee} & W \sqcup \{\text{‘does not exist’}\} \\ \wr\| & & \wr\| \\ 2^{\mathrm{Ver}} & \xrightarrow{\cap} & \mathrm{Ver} \sqcup \{\text{‘not a Verma module’}\} \end{array}$$

commute (where all isomorphisms are induced by $\Delta : w \mapsto \Delta_w$)?

For $\mathfrak{sl}_n$, the answer is **yes** by the previous question.

In general, the answer is **no**: there are $w = x \vee y$ while $\Delta_x \cap \Delta_y$ is not a Verma module.

Question 2. For which subsets of W (and of Ver) does $\vee$ and $\cap$ agree?

Join-irreducible and bigrassmannian elements

Definition.

- A simple element $s \in S$ is called a *left (resp., right) descent* of $w \in W$ if $sw < w$ (resp., $ws < w$).
- An element $w \in W$ is called *bigrassmannian* if it has a unique left descent and a unique right descent.
- An element in $x \in W$ is called *join-irreducible* if x is not the join of some $U \subset W \backslash \{x\}$.

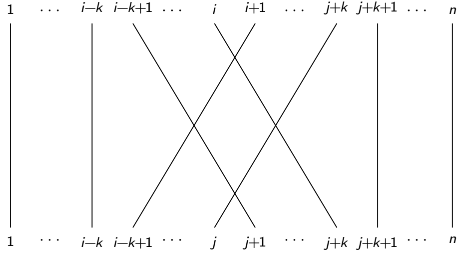
Lascoux-Schützenberger: For S_n, the join-irreducibles are exactly the bigrassmannians.

Geck-Kim: Every join-irreducible element in $(W, \leqslant)$ is bigrassmannian. The converse is not true for most Weyl groups other than S_n.

Well-known: Any element $w \in W$ is the join of some join-irreducible elements.

Bigrassmannians in S_n

A bigrassmannian permutation is of the form



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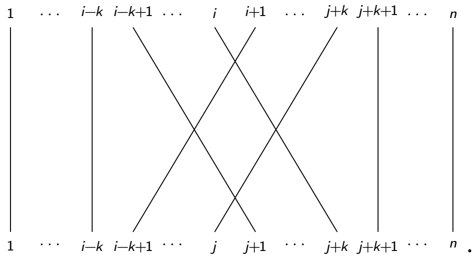
These!



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Well-known: Any element $w \in W$ is the join of some join-irreducible elements.

Geck-Kim's enumeration (1997)

In type B_n ($\mathfrak{g} = \mathfrak{so}_{2n+1}$), the number g_n of bigrassmannians and the number b_n of join-irreducibles are given as

$$\sum_{n\geq 1} g_n z^{n-1} = \frac{1 - z + z^2}{(z-1)^5} = 1 + 6z + 19z^2 + 45z^3 + 90z^4 + 161z^5 + \cdots .$$

$$\sum_{n\geq 1} b_n z^{n-1} = \frac{(z+1)^2}{(1-z)^4} = 1 + 6z + 19z^2 + 44z^3 + 85z^4 + 146z^5 + \cdots .$$

Geck-Kim's computation (1997)

TABLE I
Base and Bi-grassmannians for Exceptional Types

(W, S)	$\#W$	$\#\mathrm{BiGr}(W)$	$\#\mathrm{Base}(W)$	'clivage'
$I_2(m)$	$2m$	$2(m-1)$	$2(m-1)$	yes
H_3	120	43	42	yes
H_4	14400	756	469	yes
G_2	12	10	10	yes
F_4	1152	108	96	no
E_6	51840	232	182	no
E_7	2903040	945	528	no
E_8	696729600	8460	2060	no

Here the set of join-irreducibles is called *base*.

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Well-known: Any element $w \in W$ is the join of some join-irreducible elements.

Our answer to Question 2

Let JI be the set of join-irreducibles in W.

Define. Given $w \in W$, let $\mathrm{JM}(w)$ be the set of maximal elements in

$\{x \in \mathrm{JI} \mid$ a left/right descent of x is a left/right descent of w, respectively$\}$.

Lemma

For each $w \in W$, we have $w = \vee \mathrm{JM}(w)$.

Conjecture [K.-Mazorchuk-Mrden 2021]

For each $w \in W$, we have

$$\Delta_w = \bigcap_{x \in \mathrm{JM}(w)} \Delta_x.$$

A different answer to Question 1?

Recall we had

$$\begin{array}{ccc} (W, \leqslant) & \xrightarrow{\simeq} & (\{\Delta_w\}, \supset) = \mathrm{Ver} \\ \cap & & \cap \\ \mathrm{Lat}(W) & \xrightarrow{\simeq} & \{\bigcap_{w \in U} \Delta_w \mid U \subset W\} \end{array}$$

in type A and not in general.

Question 3. Is there a different characterization of $\{\bigcap_{w \in U} \Delta_w \mid U \subset W\}$ in type A, which yields new posets in other types?

An answer (which is 'yes') will come from the following

Observation

The socle of $\Delta_e / \bigcap_{x \in U} \Delta_x$ is the sum of the socles of Δ_e / Δ_x over $x \in U$.

The penultimate cell $\mathcal{J}$ in W

Kazhdan-Lusztig defines a preorder $\leqslant_J$ on the set W (in terms of the KL basis).

The cells with respect to $\leqslant_J$ are called the *KL two-sided cells*, or *J-cells*.

Basic fact. The J-cells in W are ordered as

$$\{e\} <_J \mathcal{J}_0 <_J \text{ “the difficult part (not linearly ordered) ” } <_J \mathcal{J} <_J \{w_0\}$$

for small cells $\mathcal{J}_0$ and $\mathcal{J}$.

The cell $\mathcal{J}_0$ consists of the elements with unique reduced expressions, and $\mathcal{J} = w_0\mathcal{J}_0 = \mathcal{J}_0 w_0$.

Example. In rank 2, e.g., S_3, we have $\mathcal{J}_0 = \mathcal{J} = W \backslash \{e, w_0\}$.

A more interesting answer to Question 1

Definition. A graded subquotient of Δ_e is a *$\mathcal{J}$-subquotient* if its simple factors are of the form L_z for $z \in \mathcal{J}$.

Theorem [K. 2021, K.-Mazorchuk-Mrden 2020]

Let $W = S_n$. Then we have an equality of sets

$$\{\bigcap_{w \in U} \Delta_w \mid U \subset W\} = \{M \subset \Delta_e \mid \mathrm{soc}(\Delta_e/M) \text{ is a } \mathcal{J}\text{-subquotient}\}.$$

In general, we have an (often strict) inclusion "$\subset$".

Question. Does this answer Question 1, i.e., is the RHS the MacNeille completion $\mathrm{Lat}(Ver)$?

Answer. No. But $\mathcal{J}$-subquotients are useful for other things!

Homological algebra

Proposition [K.-Mazorchuk-Mrden 2020]

For any W and $w \in W$, the socle of Δ_e/Δ_w is a $\mathcal{J}$-subquotient.

We (2020) compute all $\operatorname{soc} \Delta_e/\Delta_w$ in type A;

We (2021) give an upper bound to $\operatorname{soc} \Delta_e/\Delta_w$ for all Weyl types as well as computing the exact multiplicities for small ranks.

The socles of Δ_e/Δ_w are of interest also because of the following

Lemma

$\mathrm{Ext}^1(L_x, \Delta_w) \cong \mathrm{Hom}(L_x, \operatorname{soc} \Delta_e/\Delta_w) \cong \mathbb{C}^{[\operatorname{soc} \Delta_e/\Delta_w : L_x]}$.

As a consequence, we computes all $\mathrm{Ext}^1(L_x, \Delta_w)$ in type A and give an upper bound to $\mathrm{Ext}^1(L_x, \Delta_w)$ for all Weyl types as well as computing the exact dimensions for small ranks.

The $\mathcal{J}$-subquotients in type A

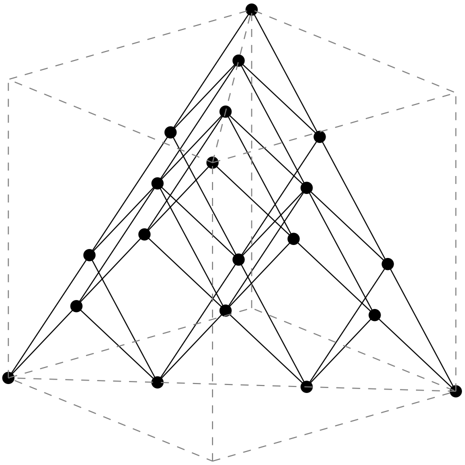

Figure: Picture of Δ_e for $\mathfrak{sl}_6$. A black dot is a simple $\mathcal{J}$-subquotient. The black dots corresponds to the join-irreducibles.

The same Verma module as in the previous slide

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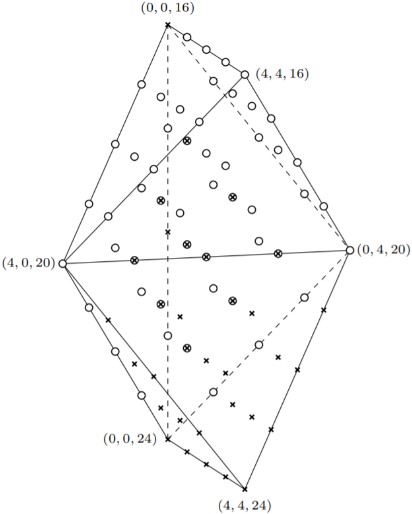
The $\mathcal{J}$-subquotients in type B



(We determine the $\mathcal{J}$-subquotients for all finite Coxeter types.)

Picturing $\Delta_w \subset \Delta_e$

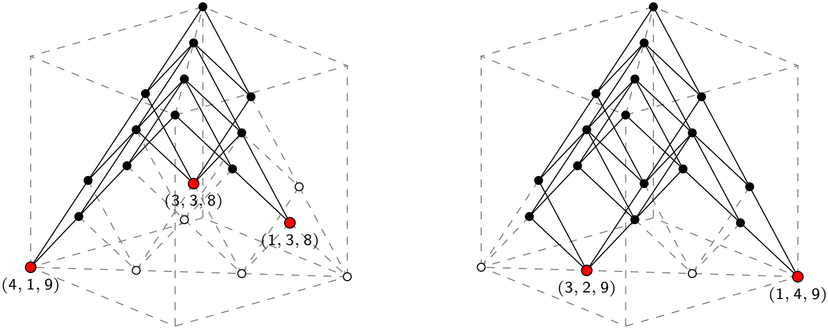
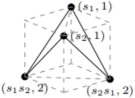


Figure: Some $\mathfrak{sl}_6$ examples. The socle of Δ_e/Δ_w (red dots) draws the boundary between Δ_e/Δ_w and Δ_w

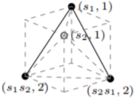
We have a complete description in type A, in a tetrahedron as above, and conjectural descriptions in general.

The S_3 example

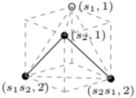
$\Delta_e =$



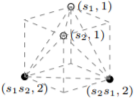
$\Delta_{s_1} =$



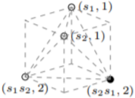
$\Delta_{s_2} =$



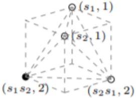
$\Delta_{s_1} \cap \Delta_{s_2} =$



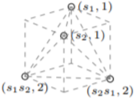
$\Delta_{s_2 s_1} =$



$\Delta_{s_1 s_2} =$



$\Delta_{w_0} =$



Homological algebra

Proposition [K.-Mazorchuk-Mrden 2020]

For any W and $w \in W$, the socle of Δ_e/Δ_w is a $\mathcal{J}$-subquotient.

We (2020) compute all $\operatorname{soc}\Delta_e/\Delta_w$ in type A;

We (2021) give an upper bound to $\operatorname{soc}\Delta_e/\Delta_w$ for all Weyl types as well as computing the exact multiplicities for small ranks.

The socles of Δ_e/Δ_w are of interest also because of the following

Lemma

$\operatorname{Ext}^1(L_x, \Delta_w) \cong \operatorname{Hom}(L_x, \operatorname{soc}\Delta_e/\Delta_w) \cong \mathbb{C}^{[\operatorname{soc}\Delta_e/\Delta_w : L_x]}$.

As a consequence, we computes all $\operatorname{Ext}^1(L_x, \Delta_w)$ in type A and give an upper bound to $\operatorname{Ext}^1(L_x, \Delta_w)$ for all Weyl types as well as computing the exact dimensions for small ranks.

Thank you!

Papers:

Bigrassmannian permutations and Verma modules (arXiv:2008.08864)

Join operation for the Bruhat order and Verma modules (arXiv:2109.01067)

Alternating sign matrices and Verma modules (arXiv:2108.07637)

More on category $\mathcal{O}$: Uppsala Algebra on YouTube

Comparing socles

Let U be a set and let $M, N, N_u \in \mathcal{O}$, for $u \in U$, with $N, N_u \subseteq M$.

1. We say that *the socle of M/N is contained in the sum of the socles of M/N_u* if
   - for each $u \in U$, we have $N \subset N_u$ (Note that this induces the map
     $$\phi_u : \mathrm{soc}(M/N) \to \mathrm{soc}(M/N_u)$$
     for each $u \in U$);
   - we have
     $$\bigcap_{u \in U} \ker \phi_u = 0.$$
2. We say that *the socle of M/N contains the sum of the socles of M/N_u* if
   - for each $u \in U$, we have $N \subseteq N_u$;
   - each ϕ_u is surjective.
3. We say that *the socle of M/N agrees with the sum of the socles of M/N_u* if it contains and is contained in the socles of M/N_u.