On the restriction of a character of $\mathfrak{S}_n$ to a Sylow p-subgroup

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Joint work with E. Giannelli

Sylow Branching Coefficients

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Frobenius Reciprocity

$$Z_\phi^\chi = [\chi\big\downarrow_P, \phi] = \left[\chi, \phi\big\uparrow^G\right]$$

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- The set of the irreducible constituents of $1_P\big\uparrow^G$ controls some local properties of G.

Giannelli, Navarro; 2018

We say that an irreducible character χ of G is **linear** if it has degree 1, i.e. $\chi(1) = 1$.

Conjecture

Let $\chi \in \mathrm{Irr}(G)$, $p|\chi(1)$.
If $\chi\downarrow_P$ has a linear constituent, then $\chi\downarrow_P$ has p linear constituents.

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- Let $\mathfrak{S}_n$ be the symmetric group on n elements and $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$.

Theorem

The conjecture just stated holds for $\mathfrak{S}_n$.
Moreover, $\chi\downarrow_{P_n}$ has a linear constituent, for each $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$.

Linear constituents

Theorem (Giannelli, Navarro; 18)

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(The situation is completely different when $p = 2$.)

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Higher degree constituents

Which irreducible character of $\mathfrak{S}_n$ has a constituent of chosen degree when restricted to its Sylow p-subgroup P_n?

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- Let $\mathrm{cd}(G) = \{ \chi(1) \mid \chi \in \mathrm{Irr}(G) \}$ be the set of the **character degrees** of a finite group G.
  Then
  $$\mathrm{cd}(P_n) = \left\{ p^k \;\middle|\; 0 \le k \le \alpha_n \right\}.$$

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Definition

Let n be a natural number. Fix $p^k \in \mathrm{cd}(P_n)$:

$$\Omega_n^k := \left\{ \chi \in \mathrm{Irr}(\mathfrak{S}_n) \,\middle|\, Z_\phi^\chi \neq 0 \text{ for some } \phi \in \mathrm{Irr}(P_n),\ \phi(1) = p^k \right\}.$$

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- The result by Giannelli and Navarro tells us that $\Omega_n^0 = \mathrm{Irr}(\mathfrak{S}_n)$, for every natural number n.

Ordinary irreducible characters

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Let χ^λ be the irreducible character of $\mathfrak{S}_n$ associated to S^λ.
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Our goal:

$$\Omega_n^k = \left\{ \chi \in \mathrm{Irr}(\mathfrak{S}_n) \,\middle|\, Z_\phi^\chi \neq 0 \text{ for some } \phi \in \mathrm{Irr}(P_n),\ \phi(1) = p^k \right\}$$

$$\Longrightarrow \Omega_n^k = \left\{ \lambda \in \mathscr{P}(n) \,\middle|\, \chi^\lambda \in \Omega_n^k \right\} \subseteq \mathscr{P}(n)$$

Combinatorial elements

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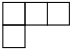
$Y(\lambda) =$ 

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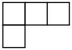
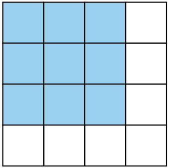
$Y(\lambda) =$ 

$\mathscr{B}_4(3)$ 

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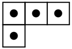
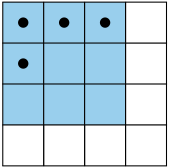
$Y(\lambda) =$ 

$\mathscr{B}_4(3)$ 

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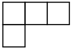
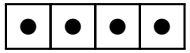
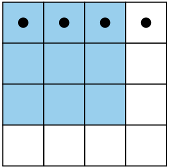
$Y(\lambda) =$ 

$Y(\mu) =$ 

$\mathscr{B}_4(3)$ 

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First main result

$$\Omega_n^k = \left\{ \lambda \in \mathscr{P}(n) \;\middle|\; Z_\phi^{\chi^\lambda} \neq 0 \text{ for some } \phi \in \mathrm{Irr}(P_n),\ \phi(1) = p^k \right\}$$

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Theorem (Giannelli, V; 2021)

Let p be an odd prime. Let $p^k \in \mathrm{cd}(P_n)$. Then there exists $t_n^k \in \mathbb{N}$ such that

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- The result of Giannelli and Navarro can be rewritten as:

$$\Omega_n^0 = \mathrm{Irr}(\mathfrak{S}_n) = \mathscr{P}(n) = \mathscr{B}_n(n) \implies t_n^0 = n, \text{ for every } n \in \mathbb{N}.$$

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- If $p = 2$, the theorem doesn't hold:
  Example: $\Omega_4^1 = \{ (3,1), (2,1,1) \}$.

Regularity of Ω_n^k when p is odd

For every $n \in \mathbb{N}$ and every k such that $p^k \in \mathrm{cd}(P_n)$, we have explicitly computed the value of t_n^k.

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Corollary

1. $\Omega_n^k \subseteq \Omega_n^j$, *for every* $j \leq k$*;*
2. $t_n^{k+1} \in \{ t_n^k - 1, t_n^k \}$.

Example: $p = 3$

$\Omega^k_{3^n}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$k = 0$	$\mathscr{B}_3(3)$	$\mathscr{B}_{3^2}(3^2)$	$\mathscr{B}_{3^3}(3^3)$	$\mathscr{B}_{3^4}(3^4)$
$k = 1$	$\emptyset$	$\mathscr{B}_{3^2}(3^2 - 1)$	$\mathscr{B}_{3^3}(3^3 - 1)$	$\mathscr{B}_{3^4}(3^4 - 1)$
$k = 2$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3 - 1)$	$\mathscr{B}_{3^4}(3^4 - 1)$
$k = 3$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3 - 2)$	$\mathscr{B}_{3^4}(3^4 - 1)$
$k = 4$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3 - 3)$	$\mathscr{B}_{3^4}(3^4 - 2)$
$k = 5$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 2)$
$k = 6$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 3)$
$k = 7$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 3)$
$k = 8$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 4)$
$k = 9$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 5)$
$k = 10$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 6)$
$k = 11$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 7)$
$k = 12$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 8)$
$k = 13$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4 - 9)$
$k = 14$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Theorem (Giannelli, V; 2021)

For every $p^k \in \mathrm{cd}(P_n)$,

$$\lim_{n\to\infty} \frac{|\Omega_n^k|}{|\mathscr{P}(n)|} = 1.$$

This means that, when n is big enough, "almost all" irreducible characters of $\mathfrak{S}_n$ has a constituent for every possible degree in the Sylow p-subgroup P_n.

Proof of the first main result

(1) Let $n = \sum_{i=1}^{r} p^{n_i}$ be the p-adic expansion of n. Then

$$P_n = P_{p^{n_1}} \times P_{p^{n_2}} \times \cdots \times P_{p^{n_r}}$$

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(2) Consider $n = p^t$. Denote by C_p a cyclic group of order p in $\mathfrak{S}_n$, then

$$P_{p^t} = \underbrace{C_p \wr \cdots \wr C_p}_{t} = P_{p^{t-1}} \wr C_p = \left(P_{p^{t-1}}\right)^{\times p} \rtimes C_p.$$

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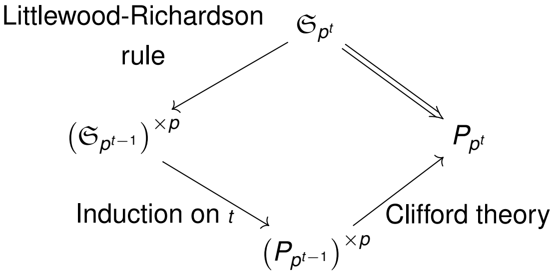
Clifford theory

$\mathrm{Irr}(P_{p^t}) = A \sqcup B$, where

- $A = \left\{ \phi_1 \times \cdots \times \phi_p \big\uparrow^{P_{p^t}} \,\middle|\, \phi_i \in \mathrm{Irr}(P_{p^{t-1}}) \text{ not all equal} \right\}$, and
- $B = \left\{ \widehat{(\phi)^{\times p}} \times \alpha \,\middle|\, \phi \in \mathrm{Irr}(P_{p^{t-1}}),\ \alpha \in \mathrm{Irr}(C_p) \right\}$, where $\widehat{(\phi)^p}$ denotes the inflation.

Sketch of the proof



Fundamental result

The following lemma was fundamental in the proof of the main result for the odd case:

Lemma

Let p be an odd prime and $\lambda \in \Omega_n^k \setminus \{ (n), (1, \dots, 1) \}$.
Then there exist at least two different constituents $\phi \neq \psi$ of $\chi^\lambda \downarrow_{P_n}$ of degree $\phi(1) = \psi(1) = p^k$.

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<u>For instance</u>: If λ is as in the above lemma, we have

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We need more:

We need **more than two** different constituent of the same degree to go on with the induction argument.
Indeed: if $\phi, \psi \in \mathrm{Irr}(P_{2^{n-1}})$ are distinct and of the same degree, then $(\phi \times \psi)\uparrow^{P_{2^n}} = (\psi \times \phi)\uparrow^{P_{2^n}}$ is a unique irreducible character of P_{2^n}.

Restriction to the hooks

Idea

Restrict our sets: let

$$\mathscr{H}(n) = \{ (n-x, 1, \ldots, 1) \mid 0 \leq x \leq n-1 \} \subseteq \mathscr{P}(n)$$

be the set of the **hooks** of n.

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Let $p = 2$ and n a natural number. From now on $P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$.
Consider:

$$\mathscr{H}_n^k := \Omega_n^k \cap \mathscr{H}(n), \text{ for } 2^k \in \mathrm{cd}(P_n), \text{ and}$$
$$\bar{\mathscr{B}}_n(t) := \mathscr{B}_n(t) \cap \mathscr{H}(n), \text{ for } t \leq n.$$

Main results when $p = 2$

Theorem

Let $n \in \mathbb{N}$. For every $\ell \leq k$ such that $2^\ell, 2^k \in \mathrm{cd}(P_n)$, we have

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- The restriction to P_n of most of the irreducible characters labelled by hook partitions admit irreducible constituents of every possible degree.

Conclusions

What we know:

p odd
- which are the linear constituents of $\chi\!\downarrow_{P_n}$;
- if there exists $\phi \in \mathrm{Irr}(P_n)$ of degree p^k among the constituents of $\chi\!\downarrow_{P_n}$;

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Open questions:

p odd

- which are the non-linear constituents of $\chi\downarrow_{P_n}$;
- what are the Sylow branching coefficients, both for the linear and the non-linear case;

$p = 2$

- what happens for λ not a hook partition;
- which are the constituents of $\chi\downarrow_{P_n}$;
- what are the Sylow branching coefficients.

Thanks for your attention!