From Klyachko models to perfect models

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Outline of the talk

- Preliminaries and conventions
- Klyachko model for $\mathbf{GL}_n(\mathbb{F}_q) \rightsquigarrow$ involution model for S_n
- Involution models for Coxeter groups $\rightsquigarrow$ Geland models for Iwahori-Hecke algebras
- Digression: another Iwahori-Hecke module on involutions and its surprising decomposition
- Canonical bases and W-graphs indexed by involutions

Model representations

In this talk, we will always work over nice fields/rings so that repns are completely reducible.

Irreducible (in context of group representations or characters) means irreducible over $\mathbb{C}$.

Write $\mathrm{Irr}(G)$ for the set of irreducible characters of a finite group G.

Some more conventions when G is a finite group:

- For a repn $\rho : H \to \mathbf{GL}(V)$ of a subgroup $H \subseteq G$, the **induced repn** is $\mathrm{Ind}_H^G(\rho)$. As a G-module the induced representation can be realized as $\mathbb{C}G \otimes_{\mathbb{C}H} V$.
- A **model** for G is a set of 1-dimensional representations $\{\lambda_i : H_i \to \mathbb{C}^\times\}$ of subgroups.
- The corresponding **model representation** is $\rho = \bigoplus_i \mathrm{Ind}_{H_i}^G(\lambda_i)$. These are precisely the G-representations that have bases in which $\rho(g)$ is a **monomial matrix** $\forall g \in G$.

> A **Gelfand model** for G is a representation with character $\sum_{\chi \in \mathrm{Irr}(G)} \chi$.
>
> That is, a Gelfand model for a finite group has a unique irreducible subrepresentation from every isomorphism class. A **Gelfand model** for a semisimple algebra is defined analogously.

Coxeter groups

We are going to talk a lot about **(finite) Coxeter systems**. A very brief refresher:

- Every Coxeter group W comes with a set of **simple generators** S that are involutions.
- Call (W, S) a **Coxeter system**. Its **length function** $\ell : W \to \{0, 1, 2, \dots\}$ counts the factors in any shortest expression for an element as a product of simple generators.
- Write $s \sim t$ if $s, t \in S$ and $st \neq ts$. (W, S) is **irreducible** if S has only one $\sim$-equiv class.

Finite irreducible **crystallographic types**: A_n, B_n/C_n, D_n, E_6, E_7, E_8, F_4, G_2

- $W_{A_{n-1}}$ = the **symmetric group** S_n of permutations of $\{1, 2, 3, \dots, n\}$.
- $W_{B_n} = W_{C_n} = n \times n$ monomial matrices with all entries in $\{-1, 0, 1\} \cong C_{S_{2n}}(w_0)$.
- W_{D_n} = subgroup of matrices W_{B_n} with even number of -1 entries.

Finite irreducible **non-crystallographic types**: H_3, H_4, $I_2(m)$ for $m \notin \{2, 3, 4, 6\}$

- $W_{H_3} = \mathrm{Alt}_5 \times S_2$.
- $W_{I_2(m)}$ = dihedral group of order $2m$. (Gives $A_1 \times A_1$, A_2, B_2, G_2 for $m = 2, 3, 4, 6$.)

Model for finite general linear and symmetric groups

Klyachko found a surprisingly simple Gelfand model for $\mathbf{GL}_n(\mathbb{F}_q)$.

Let $\mathbf{UT}_n(\mathbb{F}_q)$ be the group of $n \times n$ unipotent upper triangular matrices over $\mathbb{F}_q$

Choose a nontrivial homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$. For each $0 \leq 2d \leq n$ let

$$H_d = \left\{ \begin{bmatrix} g & h \\ 0 & x \end{bmatrix} : g \in \mathbf{Sp}_d(\mathbb{F}_q),\ x \in \mathbf{UT}_{n-2d}(\mathbb{F}_q) \right\} \text{ and } \lambda_d\left(\begin{bmatrix} g & h \\ 0 & x \end{bmatrix}\right) = \psi\left(\textstyle\sum_{i=1}^{n-2d-1} x_{i,i+1}\right).$$

Theorem (Klyachko, 1984). $\sum_{d=0}^{\lfloor n/2 \rfloor} \operatorname{Ind}_{H_d}^{\mathbf{GL}_n(\mathbb{F}_q)}(\lambda_d) = \sum_{\chi \in \operatorname{Irr}(\mathbf{GL}_n(\mathbb{F}_q))} \chi.$

Inglis–Richardson–Saxl observed that a similar Gelfand model exists for symmetric group S_n.

Define $W_{B_n} = C_{S_{2n}}(w_0) = \operatorname{Weyl}(\mathbf{Sp}_n)$ for $w_0 = 2n \cdots 321$. For each $0 \leq 2d \leq n$ redefine

$$H_d = W_{B_d} \times S_{n-2d}, \quad \lambda_d(w_{2d} \times \sigma_{n-2d}) = \operatorname{sgn}(\sigma_{n-2d}), \quad \tilde{\lambda}_d(w_{2d} \times \sigma_{n-2d}) = \operatorname{sgn}(w_{2d}).$$

Theorem (IRS, 1990). $\sum_{d=0}^{\lfloor n/2 \rfloor} \operatorname{Ind}_{H_d}^{S_n}(\lambda_d) = \sum_{d=0}^{\lfloor n/2 \rfloor} \operatorname{Ind}_{H_d}^{S_n}(\tilde{\lambda}_d) = \sum_{\chi \in \operatorname{Irr}(S_n)} \chi.$

Involution models and generalized involution models

Both $\{\lambda_d : H_d \to \mathbb{C}\}$ and $\{\tilde{\lambda}_d : H_d \to \mathbb{C}\}$ are **involution models** for S_n: the subgroups H_d are centralizers of the distinct conjugacy classes of **involutions** $w = w^{-1} \in S_n$, and every irreducible representation of S_n appears once as constituent of induced 1-dim repns.

Natural to ask which finite Coxeter groups W have involution models, since:

- involution model for $W \leftrightarrow$ Gelfand model defined on span of $\{w = w^{-1} \in W\}$,
- all irreducible representations of finite Coxeter groups are realizable over $\mathbb{R}$,
- for finite groups G with this property $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{g = g^{-1} \in G\}|$,

Theorem (Baddeley, 1990s; Vinroot, 2008). A finite Coxeter group W has an involution model iff all of its irreducible factors are of type A_n, B_n, D_{2n+1}, H_3, or $I_2(m)$.

Theorem (M.–Caselli, 2014). A finite Coxeter group W has inv. model iff it has a **generalized involution model (GIM)**, and $G(r,p,n)$ has a GIM iff $G(r,p,n) \cong G(r,n)/\mathbb{Z}_p$.

Example: $W_{D_{2n}}$ has no GIM since $W_{D_n} = G(2,2,n) \cong G(2,n)/\mathbb{Z}_2 = W_{B_n}/\{\pm 1\}$ iff n odd.

Gelfand models for the symmetric group

Let $s_i = (i, i+1) \in S_n$ so that S_n is generated by $S = \{s_1, s_2, \dots, s_{n-1}\}$.

IRS involution models $\leftrightarrow$ two Gelfand S_n-repns spanned by $\{M_w : w = w^{-1} \in S_n\}$ with

$$H_s M_w = \begin{cases} M_{sws} & \text{if } s \in \mathrm{Asc}^<(w) \sqcup \mathrm{Des}^<(w) \\ \pm M_w & \text{if } s \in \mathrm{Des}^=(w) \\ \mp M_w & \text{if } s \in \mathrm{Asc}^=(w) \end{cases} \qquad \text{for } s \in S,\ w = w^{-1} \in S_n$$

for certain **strict/weak ascent/descent sets** $\mathrm{Asc}^<(w)$, $\mathrm{Des}^<(w)$, $\mathrm{Des}^=(w)$ and $\mathrm{Asc}^=(w)$.

Extend $w = w^{-1} \in S_n$ fixing $i_1 < \cdots < i_k$ to $\underline{w} = \underline{w}^{-1} \in S_{n+k}$ by $\underline{w}(i_j) := n + j$. Then

$$\mathrm{Des}^=(w) = \{s \in S : \underline{w}s\underline{w} = s\},$$
$$\mathrm{Asc}^=(w) = \{s \in S : \underline{w}s\underline{w} \in \{s_{n+1}, s_{n+2}, \dots, s_{n+k-1}\}\},$$

$$\mathrm{Des}^<(w) = \{s_i \in S : w(i) > w(i+1)\} - \mathrm{Des}^=(w) \sqcup \mathrm{Asc}^=(w),$$
$$\mathrm{Asc}^<(w) = \{s_i \in S : w(i) < w(i+1)\} - \mathrm{Des}^=(w) \sqcup \mathrm{Asc}^=(w).$$

Involution models for Iwahori-Hecke algebras

Let (W, S) be a Coxeter system with length function ℓ. If $W = S_n$ then $S = \{s_1, \dots, s_{n-1}\}$. The corresponding **Iwahori-Hecke algebra** $\mathcal{H}(W) = \mathbb{Q}[x^{\pm 1}]$-span$\{H_w : w \in W\}$ has

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w) \\ H_{sw} + (x - x^{-1})H_w & \text{if } \ell(sw) < \ell(w) \end{cases} \quad \text{for } w \in W \text{ and } s \in S.$$

Theorem (Adin–Postnikov–Roichman, 2008). Let $W = S_n$. For each sign $\alpha \in \{\pm 1\}$, there is a Gelfand $\mathcal{H}(W)$-module $\mathbf{GMod}^\alpha$ with basis $\{M_w : w = w^{-1} \in W\}$ such that

$$H_s M_w = \begin{cases} M_{sws} & \text{if } s \in \mathrm{Asc}^<(w) \\ M_{sws} + (x - x^{-1})M_w & \text{if } s \in \mathrm{Des}^<(w) \\ x_\alpha M_w & \text{if } s \in \mathrm{Des}^=(w) \\ -x_\alpha^{-1} M_w & \text{if } s \in \mathrm{Asc}^=(w) \end{cases} \quad \text{for } s \in S,\ w = w^{-1}, \text{ where } x_\alpha := \alpha x^\alpha.$$

Theorem (M.–Zhang, 2022). Same result holds if W is any finite Coxeter group with an involution model, for certain explicit sets $\mathrm{Asc}^<(w)$, $\mathrm{Des}^<(w)$, $\mathrm{Des}^=(w)$, $\mathrm{Asc}^=(w)$.

Perfect models

The Gelfand models $\mathbf{GMod}^+$ and $\mathbf{GMod}^-$ are not produced directly from an involution model. Instead, they are features of a more technical construction called a **perfect model**.

> **Theorem (M.–Zhang, 2022).** A finite Coxeter group has an involution model if and only if it has a perfect model.

Any perfect model $\mathcal{P}$ determines a pair of Gelfand $\mathcal{H}(W)$-models analogous to $\mathbf{GMod}^{\pm}$.

These modules then gives rise to a pair of ***W*-graphs** $\Gamma^+_{\mathcal{P}}$ and $\Gamma^-_{\mathcal{P}}$ which will be discussed later.

There is a notion of **equivalence**, with $\mathcal{P}_1 \equiv \mathcal{P}_2 \Rightarrow \textbf{cells}\left(\Gamma^+_{\mathcal{P}_1} \sqcup \Gamma^-_{\mathcal{P}_1}\right) \cong \textbf{cells}\left(\Gamma^+_{\mathcal{P}_2} \sqcup \Gamma^-_{\mathcal{P}_2}\right)$.

> **Theorem (M.–Zhang, 2022).** Outside rank 3, and ignoring a trivial family of exceptions in type B_n, each irreducible W has at most one equivalence class of perfect models.

To find perfect model $\mathcal{P}$: choose $J \subseteq S$, "**perfect**" $w = w^{-1} \in W_J$, repn $\sigma : W_J \to \{\pm 1\}$ so

$$\textstyle\sum_{(J,w,\sigma)\in\mathcal{P}} \mathrm{Ind}_{C_{W_J}(w)}^{W} \mathrm{Res}_{C_{W_J}(w)}^{W_J}(\sigma) = \sum_{\chi\in\mathrm{Irr}(W)} \chi \quad \left[\text{perfect} \Leftrightarrow (wt)^4 = 1\ \forall t \in T\right].$$

Bar operators and canonical bases

An **antilinear map** $L : \mathcal{H}(W) \to \mathcal{H}(W)$ is a $\mathbb{Q}$-linear map with $L(x^n h) = x^{-n} L(h)$.
The **bar involution** of $\mathcal{H}(W)$ is the antilinear ring automorphism $h \mapsto \overline{h}$ with $\overline{H_w} = (H_{w^{-1}})^{-1}$.

Theorem (Kazhdan–Lusztig, 1979). $\mathcal{H}(W)$ has a unique basis $\{\underline{H}_w\}_{w \in W}$ satisfying

$$\underline{H}_w = \overline{\underline{H}_w} \in H_w + \textstyle\sum_{\ell(y)<\ell(w)} x^{-1}\mathbb{Z}[x^{-1}] H_y.$$

Assume W is finite & has involution model $\rightsquigarrow$ Gelfand $\mathcal{H}(W)$-models $\mathbf{GMod}^{\pm}$ are defined.

Theorem (M.-Zhang, 2022). $\mathbf{GMod}^{\pm}$ has an antilinear **bar involution** $m \mapsto \overline{m}$ with

$$\overline{M_w} = M_w \text{ if } w = w^{-1} \text{ has } \mathrm{Des}^{<}(w) = \varnothing \quad \text{and} \quad \overline{hm} = \overline{h}\,\overline{m} \text{ for all } h \in \mathcal{H}(W).$$

$\mathbf{GMod}^{\pm}$ has a unique **canonical basis** $\{\underline{M}_w\}_{w=w^{-1}}$ with (for a certain **height map** ht)

$$\underline{M}_w = \overline{\underline{M}_w} \in M_w + \textstyle\sum_{\mathrm{ht}(y)<\mathrm{ht}(w)} x^{-1}\mathbb{Z}[x^{-1}] M_y. \quad \left[\text{e.g., if } W = S_n \text{ then } \mathrm{ht}(w) = \ell(\underline{w})\right]$$

Elias–Williamson (2013): $\underline{H}_w \in \mathbb{N}[x^{-1}]\text{-span}\{H_y : y \in W\}$. No such general positivity for $\underline{M}_w$.

Involution representations for all Iwahori-Hecke algebras

There is another way of lifting the IRS involution representations to Iwahori-Hecke algebra.

Theorem (Lusztig, 2012). Let (W, S) be any Coxeter system. Choose $\alpha \in \{\pm 1\}$. Then there is an $\mathcal{H}(W)$-module $\textbf{Invol}^\alpha$ with basis $\{I_w : w = w^{-1} \in W\}$ such that

$$H_s I_w = \begin{cases} I_{sws} & \text{if } sw \neq ws > w \\ I_{sws} + (x - x^{-1}) I_w & \text{if } sw \neq ws < w \\ (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) I_{sw} + \alpha I_w & \text{if } sw = ws > w \\ \alpha (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) I_{sw} + (x - \alpha - x^{-1}) I_w & \text{if } sw = ws < w \end{cases}$$

for $s \in S,\ w = w^{-1} \in W$; here $v < w$ means $\ell(v) < \ell(w)$.

When $x = 1$ we have $\textbf{Invol}^\alpha \cong \textbf{Invol}^{-\alpha} \otimes \mathrm{sgn}$ as W-representations.

Theorem (M., 2013). For finite W, these W-representations are Gelfand models if and only if all irreducible factors of (W, S) are of type A_n, H_3, or $I_2(m)$ with m odd.

Natural to ask what is the irreducible decomposition of $\textbf{Invol}^+$ and $\textbf{Invol}^-$ in the typical case when they are not Gelfand models. The answer is known for finite W, and sort of amazing.

Unipotent characters, formally

Lusztig attaches to each finite Coxeter group W a set of **unipotent characters** $\mathrm{Uch}(W)$.

We consider $\Phi \in \mathrm{Uch}(W)$ to be a formal object with 3 properties:

- $\mathrm{FakeDeg}(\Phi) \in \mathbb{N}[x]$, called the **fake degree**.
- $\mathrm{Deg}(\Phi) \in \mathbb{R}[x]$, called the **generic degree**.
- $\mathrm{Eig}(\Phi) \in \mathbb{C}^{\times}$, called the **Frobenius eigenvalue**.

There is always an inclusion $\mathrm{Irr}(W) \subset \mathrm{Uch}(W)$, which is equality only in type A.

- $\mathrm{FakeDeg}(\Phi) = 0$ for $\Phi \in \mathrm{Uch}(W) \setminus \mathrm{Irr}(W)$.
- $\mathrm{FakeDeg}(\Phi)$ is graded multiplicity of $\Phi \in \mathrm{Irr}(W)$ in coinvariant algebra of W.
- $\mathrm{Deg}(\Phi)|_{x=1} = \Phi(1)$ and $\mathrm{Eig}(\Phi) = 1$ for all $\Phi \in \mathrm{Irr}(W) \subset \mathrm{Uch}(W)$.

$\mathrm{Uch}(W)$ has further a decomposition into disjoint **families**.

For crystallographic types, $\mathrm{Uch}(W)$ is the set of irreducible characters in a finite group of Lie type G not orthogonal to all **Deligne-Lusztig generalized characters** R_ψ for $\psi \in \mathrm{Irr}(W)$.

Fourier transform on unipotent characters

Each W has an involution **FT** $\in$ **GL** $(\{\text{maps } \mathrm{Uch}(W) \to \mathbb{R}\})$ called its **Fourier transform**.

FT is a real matrix with rows/columns indexed by $\mathrm{Uch}(W)$. **FT** $= 1$ in type A.

For (W, S) crystallographic, **FT** is essentially matrix of scalar products $\langle \Phi, R_\psi \rangle$.

Distinguished (almost determining) properties of **FT** in all types:

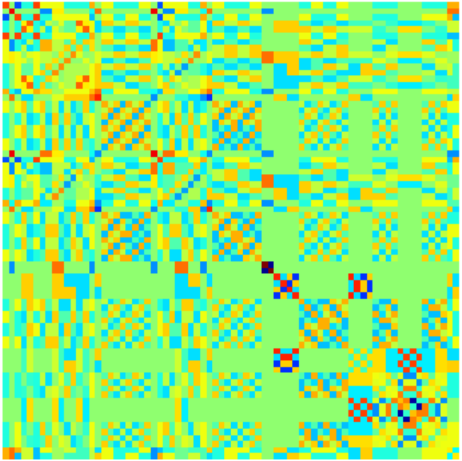
- **FT** sends fake degrees of $\mathrm{Uch}(W)$ to (a certain permutation of) its generic degrees.
- **FT** is block diagonal with respect to the division of $\mathrm{Uch}(W)$ into families.
- **FT** fixes vector of irreducible multiplicities of **left cell representations** of W.
- **FT** and diagonal matrix of Frobenius eigenvalues determine a "**fusion datum**."

Even in crystallographic types, Lusztig's original definition of **FT** is heuristic.

In non-crystallographic types, listed properties determine **FT** except on two large families of unipotent characters of size 74 in type H_4 and size $\lfloor \frac{m}{2} \rfloor \lceil \frac{m}{2} \rceil$ in type $I_2(m+2)$.

Matrices for these families were found experimentally by Malle (1994) and Lusztig (1994).

Malle's 74×74 Fourier transform matrix block in type H_4



An amazing decomposition

Recall that we have two $\mathcal{H}(W)$-modules **Invol**$^\pm$ spanned by $\{I_w : w = w^{-1} \in W\}$. When we specialize $x = 1$ these become W-representations. These turn out to be isomorphic.

Building off and summarizing prior work of Casselman, Geck, Kottwitz, and Lusztig:

Theorem (M., 2013). There is a unique function $\varepsilon : \mathrm{Uch}(W) \to \{-1, 0, 1\}$ such that

(a) $\varepsilon(\Phi) = 0$ if and only if $\mathrm{Eig}(\Phi) \not\subset \mathbb{R}$,

(b) **FT**(ε) gives the multiplicities in irreducible decomposition of **Invol**$^\pm$ when $x = 1$.

And when (W, S) is crytallographic, the map ε is exactly the **Frobenius-Schur indicator**

$$\varepsilon(\Phi) = \frac{1}{|G|} \sum_{g \in G} \Phi(g^2) = \begin{cases} 1 & \text{if } \Phi \text{ is character of a representation defined over } \mathbb{R} \\ 0 & \text{if } \Phi \text{ not real-valued} \\ -1 & \text{otherwise} \end{cases}$$

There are only two unipotent characters Φ with $\varepsilon(\Phi) = -1$, in type H_4 only.

Examples in classical types

Let $\mathrm{ch}(\mathbf{Invol}^{\pm})$ be the character of the isomorphic W-modules $\mathbf{Invol}^{+}$ or $\mathbf{Invol}^{-}$ when $x = 1$.

- In type A_n one has $\mathrm{ch}(\mathbf{Invol}^{\pm}) = \sum_{\lambda \vdash n+1} \chi^{\lambda} = \mathrm{ch}(\mathbf{GMod}^{\pm})$.
- In type B_n/C_n one has

$$\mathrm{ch}(\mathbf{Invol}^{\pm}) = \sum_{(\lambda,\mu)\vdash n} 2^{d(\lambda,\mu)} \chi^{(\lambda,\mu)}$$

  where in the sum it is required that $\mu_i \leq \lambda_i + 1$ and $\lambda_i^{\top} \leq \mu_i^{\top} + 1$ for all i.
- In type D_n one has

$$\mathrm{ch}(\mathbf{Invol}^{\pm}) = \sum_{\lambda \vdash \frac{n}{2}} \left(\chi^{\{\lambda\},1} + \chi^{\{\lambda\},2}\right) + \sum_{(\lambda,\mu)\vdash n} 2^{e(\lambda,\mu)} \chi^{\{\lambda,\mu\}}$$

  where in second sum $\lambda \subsetneq \mu$ and skew diagram $\mu \setminus \lambda$ must contain no 2×2 squares.

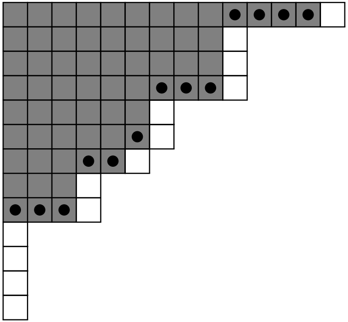
Here $d(\lambda,\mu)$ and $e(\lambda,\mu)$ are certain combinatorially defined nonnegative integers.

For example: $e(\lambda,\mu)$ is the number of connected components of skew diagram $\mu \setminus \lambda$ minus one.

Pictures of constituents of $\mathrm{ch}(\mathbf{Invol}^{\pm})$

type B_n/C_n:

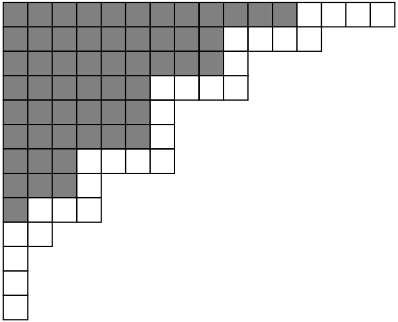
$\mu_i \leq \lambda_i + 1$ and $\lambda_i^\top \leq \mu_i^\top + 1$ means:



$\lambda =$ grey, μ formed by adding $\square$ or deleting ■ (grey square with dot)

type D_n:

$\mu \setminus \lambda$ has no 2×2 squares means:



$\lambda =$ grey, μ formed by adding $\square$'s

Two more canonical bases

$\mathcal{H}(W)$ has two 1-dim representations, generated by $\sum_{w\in W}(\alpha x^{\alpha})^{\ell(w)} w \in \mathcal{H}(W)$ for $\alpha = \pm 1$.

Theorem (Lusztig, 2014). $\mathbf{Invol}^{\alpha}$ is generated by $\sum_{w\in W}(\alpha x^{\frac{\alpha}{2}})^{\ell(w)} w$ for $\alpha = \pm 1$.

Theorem (Lusztig, 2012). Each $\mathbf{Invol}^{\pm}$ has an antilinear **bar involution** $m \mapsto \overline{m}$ with

$$\overline{I_1} = I_1 \quad \text{and} \quad \overline{hm} = \overline{h}\,\overline{m} \text{ for all } h \in \mathcal{H}(W) \text{ and } m \in \mathbf{Invol}^{\pm}.$$

Also each $\mathbf{Invol}^{\pm}$ has a unique **canonical basis** $\{\underline{I}_w\}_{w=w^{-1}}$ with

$$\underline{I}_w = \overline{\underline{I}_w} \in I_w + \sum_{\ell(y)<\ell(w)} x^{-\frac{1}{2}}\mathbb{Z}[x^{-\frac{1}{2}}] I_y.$$

As with $\underline{M}_w$, coefficients of $\underline{I}_w$ in standard basis $\{I_y : y = y^{-1} \in W\}$ not always positive.

Now we have the **Kazhdan–Lusztig basis** $\{\underline{H}_w\}$ for $\mathcal{H}(W)$, viewed as a left and right module. Also have canonical bases $\{\underline{M}_w\}$ for $\mathbf{GMod}^{+}$ and $\mathbf{GMod}^{-}$, and $\{\underline{I}_w\}$ for $\mathbf{Invol}^{+}$ and $\mathbf{Invol}^{-}$. What can one do with all of these constructions?

W-graphs in principle

Suppose $\mathcal{A}$ is an R-algebra with generators $\{a_s\}_{s\in S}$ and $\mathcal{B}$ is an $\mathcal{A}$-module with basis $\{b_v\}_{v\in V}$. Create a directed graph Γ with vertex set V and edges $v \xrightarrow[s]{c(v,w)} w$ whenever

$$a_s b_v = \sum_{w\in V} c(v,w) b_w \quad \text{and} \quad 0 \neq c(v,w) \in R.$$

Observations. We can recover $\mathcal{B}$ from Γ, and we can try to decompose $\mathcal{B}$ using Γ:

- A **cell** in Γ is a strongly connected component.
- Cells don't span literal subrepns of $\mathcal{B}$, but form vertices in a directed acyclic graph.
- This DAG defines a filtration of $\mathcal{B}$, in which each cell spans a successive quotient.
- When completely reducible, $\mathcal{B}$ is direct sum of these quotient **cell representations**.

This talk: a **W-graph** means an instance of Γ for $\mathcal{A} = \mathcal{H}(W)$ with generators $\{H_s : s \in S\}$.

In literature, "W-graph" has more specific meaning: refers to Γ's that determine $\mathcal{B}$ even if we remove all s-labels from edges, as long as vertices remember a form of "descent set."

Standard basis W-graphs: boring representations, interesting graphs

Let $\mathcal{A} = \mathcal{H}(W) = \langle H_s : s \in S \rangle$ and suppose $\mathcal{B} = \mathcal{H}(W)$ or $\mathbf{GMod}^{\pm}$ or $\mathbf{Invol}^{\pm}$.

Take $\{b_v\}_{v \in V}$ to be the **standard bases** $\{H_w\}_{w \in W}$ or $\{M_w\}_{w=w^{-1}}$ or $\{I_w\}_{w=w^{-1}}$.

Resulting W-graphs Γ are boring for representation theory:

- Every edge is **bidirected**: if $v \to w$ is an edge then so is $w \to v$ (for some labels).
- Every connected component is strongly connected: one cell if $\mathcal{B} = \mathcal{H}(W)$ or $\mathbf{Invol}^{\pm}$.
- If $\mathcal{B} = \mathbf{GMod}^{\pm}$ then # of cells is number of conjugacy classes of involutions in W.

But interesting for combinatorics:

- Form $\overrightarrow{\Gamma}$ from Γ by retaining only edges $v \underset{s}{\to} w$ with $\ell(v) < \ell(w)$ or $\mathrm{ht}(v) < \mathrm{ht}(w)$.
- If $\mathcal{B} = \mathcal{H}(W)$ then $\overrightarrow{\Gamma}$ is **left weak order** lattice for W.
- If $\mathcal{B} = \mathbf{Invol}^{\pm}$, $W = S_n$ then $\overrightarrow{\Gamma}$ is **weak order** on $\mathbf{O}_n$-orbit closures in $\mathbf{Fl}_n$.
- If $\mathcal{B} = \mathbf{GMod}^{\pm}$, $W = S_{2n}$ then $\overrightarrow{\Gamma} \leftrightarrow$ **weak order** on $\mathbf{Sp}_n$-orbit closures in $\mathbf{Fl}_{2n}$.

Maximal chains in standard basis W-graphs

Write $\vec{\Gamma}_{\mathcal{H}}$, $\vec{\Gamma}_{\textbf{Invol}}$, $\vec{\Gamma}_{\textbf{GMod}}$ for $\vec{\Gamma}$ when $\mathcal{B} = \mathcal{H}(W)$, $\textbf{Invol}^{\pm}$, $\textbf{GMod}^{\pm}$. (Same for either $\pm$)

Maximal chains in $\vec{\Gamma}_{\mathcal{H}}$ correspond to **reduced words** for longest element $w_0 \in W$.

- **Stanley (1984)**: if $W = S_n$ then # of maximal chains in $\vec{\Gamma}_{\mathcal{H}}$ is # of **standard Young tableaux** of "staircase shape" $(n-1, n-2, n-3, \dots)$.
- **M.–Pawlowski (2018)**: in type B_n this is also # of maximal chains in $\vec{\Gamma}_{\textbf{Invol}}$.
- **Hamaker–M.–Pawlowski (2015)**: if $W = S_n$ then # of maximal chains in $\vec{\Gamma}_{\textbf{Invol}}$ is # of **standard shifted tableaux** of shape $(n-1, n-3, n-5, \dots)$. This is also the # of maximal chains in component of w_0 in $\vec{\Gamma}_{\textbf{GMod}}$ if n is odd and $W = S_{n+1}$.
- **Conjecture (M.–Pawlowski, 2018)**: in type D_n the # maximal chains in $\vec{\Gamma}_{\textbf{Invol}}$ is # of **standard Young tableaux** of shape $(n-1, n-2, \dots, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \dots, 2, 1)$.

Many stronger results for **fundamental quasisymmetric descent generating functions** of maximal chains: these are always symmetric, Schur positive, Schur P-positive, etc.

Canonical basis W-graphs: interesting representations, but mysterious

Now suppose instead $\{b_v\}_{v\in V}$ is **canonical basis** $\{\underline{H}_w\}_{w\in W}$, $\{\underline{M}_w\}_{w=w^{-1}}$, or $\{\underline{I}_w\}_{w=w^{-1}}$.

- Let Γ_L and Γ_R be resulting W-graphs when $\mathcal{B} = \mathcal{H}(W)$ as left module or right module.
- Write $\Gamma^+_{\textbf{GMod}}$, $\Gamma^-_{\textbf{GMod}}$, $\Gamma^+_{\textbf{Invol}}$, $\Gamma^-_{\textbf{Invol}}$ for Γ when $\mathcal{B} = \textbf{GMod}^+$, $\textbf{GMod}^-$, $\textbf{Invol}^+$, or $\textbf{Invol}^-$.

Unlike in standard basis case, no automatic relationship $\Gamma^+_{\textbf{GMod}} \leftrightarrow \Gamma^-_{\textbf{GMod}}$ or $\Gamma^+_{\textbf{Invol}} \leftrightarrow \Gamma^-_{\textbf{Invol}}$.

The W-graphs Γ_L and Γ_R are the classical **left and right Kazhdan-Lusztig *W*-graphs**.
Their cells are often referred to simply as the **left cells** and **right cells** in W.

Theorem (Kazhdan–Lusztig, 1979). Assume $W = S_n$.

- Then each left and right cell representation is irreducible.
- In fact, each left/right cell is a **molecule** (connected by bidirected edges).
- Moreover if $w \xrightarrow{\text{RSK}} (P_{\text{RSK}}(w), Q_{\text{RSK}}(w))$ is the **RSK correspondence** then the left (resp. right) cells are the subsets where Q_{RSK} (resp. P_{RSK}) is constant.

Cells in Gelfand models and involution modules

Some things are known about cells in $\Gamma^+_{\mathbf{GMod}}$ and $\Gamma^-_{\mathbf{GMod}}$ when W has type A_n, B_n, D_{2n+1}:

Theorem (M.–Zhang, 2022). Assume $W = S_n$ is of type A.

- The **molecules** in $\Gamma^+_{\mathbf{GMod}}$ are classified by $P_{\mathsf{RSK}}(w) = Q_{\mathsf{RSK}}(w)$ for $w = w^{-1}$.
- The **molecules** in $\Gamma^-_{\mathbf{GMod}}$ are classified by a novel RSK-like insertion algorithm.

Conjecture. In type A all cells in $\Gamma^\pm_{\mathbf{GMod}}$ are molecules and all cell repns are irreducible.

Neither property is true in other classical types. However:

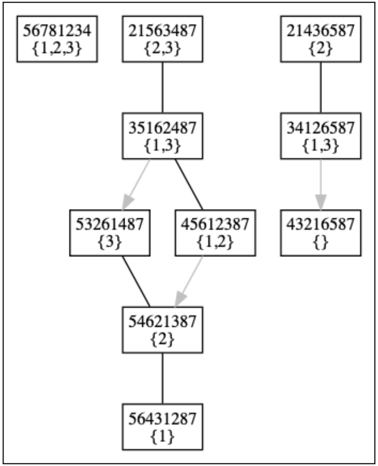
Theorem (M.–Zhang, 2022). For types B_n and D_{2n+1}, $\Gamma^+_{\mathbf{GMod}}$ and $\Gamma^-_{\mathbf{GMod}}$ are **dual**: one graph is obtained from the other by reversing all edges. This is not true in type A_n.

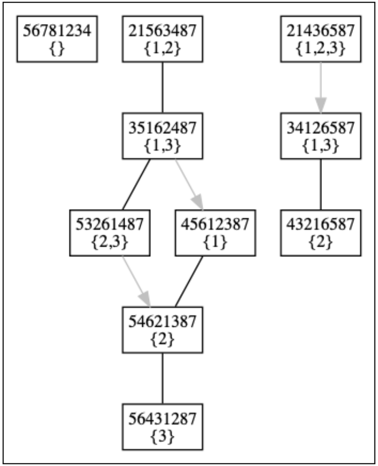
Theorem (Lusztig, 2012). If $W = S_n$ then every cell repn in $\Gamma^+_{\mathbf{Invol}}$ is irreducible.

Proof is very indirect, more concrete argument is desired! Nothing seems known about $\Gamma^-_{\mathbf{Invol}}$.

Gelfand model W-graphs for $W = S_4 = W_{A_3}$

$\Gamma^+_{\mathbf{GMod}} \cong$

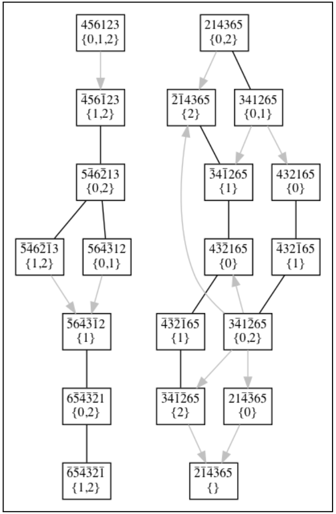


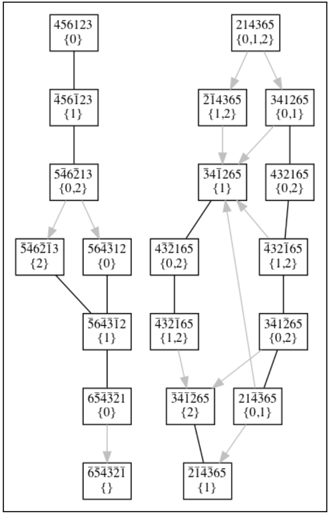


$\cong \Gamma^-_{\mathbf{GMod}}$

Gelfand model W-graphs for $W = W_{B_3} = W_{C_3}$

$\Gamma^{+}_{\mathsf{GMod}} \cong$

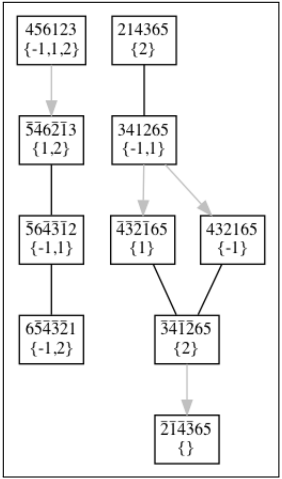


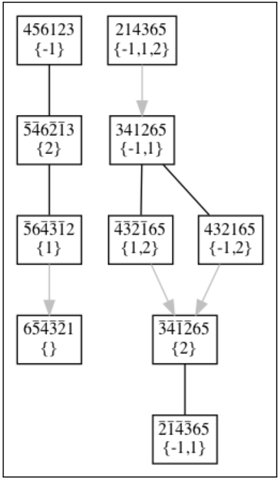


$\cong \Gamma^{-}_{\mathsf{GMod}}$

Gelfand model W-graphs for $W = W_{D_3}$

$\Gamma^{+}_{\mathbf{GMod}} \cong$





$\cong \Gamma^{-}_{\mathbf{GMod}}$

Thanks for listening!